

# Cayley graphs on groups with commutator subgroup of order $2p$ are hamiltonian

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## Abstract

We show that if  $G$  is a finite group whose commutator subgroup  $[G, G]$  has order  $2p$ , where  $p$  is an odd prime, then every connected Cayley graph on  $G$  has a hamiltonian cycle.

*Keywords:* Cayley graph, hamiltonian cycle, commutator subgroup

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## 1 Introduction

Let  $G$  be a finite group. It is easy to show that if  $G$  is abelian (and  $|G| > 2$ ), then every connected Cayley graph on  $G$  has a hamiltonian cycle. (See Definition 2.1 for the definition of the term *Cayley graph*.) To generalize this observation, one can try to prove the same conclusion for groups that are close to being abelian. Since a group is abelian precisely when its commutator subgroup is trivial, it is therefore natural to try to find a hamiltonian cycle when the commutator subgroup of  $G$  is close to being trivial. The following theorem, which was proved in a series of papers, is a well-known result along these lines.

**Theorem 1.1** (Marušič [13], Durnberger [4, 5], 1983–1985). *If the commutator subgroup  $[G, G]$  of  $G$  has prime order, then every connected Cayley graph on  $G$  has a hamiltonian cycle.*

D. Marušič (personal communication) suggested more than thirty years ago that it should be possible to replace the prime with a product  $pq$  of two distinct primes:

**Research Problem 1.2** (D. Marušič, personal communication, 1985). *Show that if the commutator subgroup of  $G$  has order  $pq$ , where  $p$  and  $q$  are two distinct primes, then every connected Cayley graph on  $G$  has a hamiltonian cycle.*

This has recently been accomplished when  $G$  is either nilpotent [8] or of odd order [14]. As another step toward the solution of this problem, we establish the special case where  $q = 2$ :

**Theorem 1.3.** *If the commutator subgroup of  $G$  has order  $2p$ , where  $p$  is an odd prime, then every connected Cayley graph on  $G$  has a hamiltonian cycle.*

See the bibliography of [12] for references to other results on hamiltonian cycles in Cayley graphs.

The proof of Theorem 1.3 is a lengthy case-by-case analysis, based on the choice of certain elements  $a$  and  $b$  of the Cayley graph's connection set (see Notation 3.3). Here is an outline of the paper:

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## 2 Some known results

We recall a few results that provide hamiltonian cycles in various Cayley graphs.

**Definition 2.1** (cf. [9, p. 34]). For any subset  $S$  of a finite group  $G$ ,  $\text{Cay}(G; S)$  is the graph whose vertex set is  $G$ , with an edge joining  $g$  to  $gs$ , for each  $g \in G$  and  $s \in S$ . This is called the *Cayley graph* of the connection set  $S$  on the group  $G$ .

**Remark 2.2.** Unlike most authors (including [9]), we do not require the connection set  $S$  to be symmetric in the definition of a Cayley graph; that is, we do not assume  $S$  is closed under inverses. This does not change the set of graphs that are considered to be Cayley graphs, because, in our notation,  $\text{Cay}(G; S) = \text{Cay}(G; S \cup S^{-1})$ , where  $S^{-1} = \{s^{-1} \mid s \in S\}$ .

**Theorem 2.3** ([12, 3, 6, 7]). *Every connected Cayley graph on  $G$  has a hamiltonian cycle if  $|G| = kp$  for some prime  $p$  and some  $k \in \mathbb{N}$  with  $1 \leq k < 32$  and  $k \neq 24$ .*

### Notation 2.4.

- The symbol  $G$  always represents a finite group.
- For  $g \in G$  and  $s_1, \dots, s_n \in S \cup S^{-1}$ , we use  $[g](s_1, \dots, s_n)$  to denote the walk in  $\text{Cay}(G; S)$  that visits (in order), the vertices

$$g, gs_1, gs_1s_2, gs_1s_2s_3, \dots, gs_1s_2 \cdots s_n.$$

We may write  $(s_1, \dots, s_n)$  for  $[e](s_1, \dots, s_n)$ .

- We use  $(s_1, \dots, s_n)^k$  to denote the concatenation of  $k$  copies of the sequence  $(s_i)_{i=1}^n$ .
- Appending  $\#$  to a sequence deletes the last term; that is,  $(s_i)_{i=1}^n \# = (s_i)_{i=1}^{n-1}$ .
- If  $W = [g](s_1, \dots, s_n)$  is a walk in  $\text{Cay}(G; S)$ , and  $h \in G$ , we use  $hW$  to denote the translate  $[hg](s_1, \dots, s_n)$ .
- When  $C$  is an oriented cycle, we use  $-C$  to denote the same cycle as  $C$ , but with the opposite orientation.
- For  $g, h \in G$ :

$$[g, h] = g^{-1}h^{-1}gh, \quad g^h = h^{-1}gh, \quad \text{and} \quad {}^hg = hgh^{-1} (= g^{h^{-1}}).$$

- We use  $G'$  to denote the commutator subgroup  $[G, G]$  of  $G$ .
- For convenience, we let  $\overline{G} = G/G'$ .
- For  $g \in G$ , we let  $\overline{g} = gG'$  be the image of  $g$  in  $\overline{G}$ .
- We use  $Z(G)$  to denote the center of  $G$ .

**Definition 2.5** (cf. [10, §2.1.3, p. 61]). Suppose

- $N$  is an abelian, normal subgroup of  $G$ , and
- $C = [Nv](s_i)_{i=1}^n$  is an (oriented) cycle in  $\text{Cay}(G/N; S)$ .

The *voltage* of  $C$  is  ${}^v(\prod_{i=1}^n s_i)$ . This is an element of  $N$ , and it may be denoted  $\Pi C$ .

We have the following straightforward observations:

[note A.1] **Lemma 2.6.** Assume the notation of Definition 2.5. Then:

1.  $\Pi C$  is determined by the oriented cycle  $C$ : it is independent of the choice of the vertex  $Nv$  of  $C$ , and of the choice of the representative  $v$  of  $Nv$ .
2.  $\Pi gC = {}^g(\Pi C)$  for all  $g \in G$ .
3.  $\Pi(-C) = (\Pi C)^{-1}$ .

**Definition 2.7.** A subset  $S$  of  $G$  is an *irredundant* generating set of  $G$  if  $S$  generates  $G$ , but no proper subset of  $S$  generates  $G$ .

**Lemma 2.8** (“Factor Group Lemma” [16, §2.2]). Suppose

- $N$  is a cyclic, normal subgroup of  $G$ ,
- $(s_i)_{i=1}^m$  is a hamiltonian cycle in  $\text{Cay}(G/N; S)$ , and
- the voltage  $\Pi(s_i)_{i=1}^m$  generates  $N$ .

Then  $(s_1, s_2, \dots, s_m)^{|N|}$  is a hamiltonian cycle in  $\text{Cay}(G; S)$ .

**Corollary 2.9** ([12, Cor. 2.11]). Suppose

- $N$  is a normal subgroup of  $G$ , such that  $|N|$  is prime,
- the image of  $S$  in  $G/N$  is an irredundant generating set of  $G/N$ ,
- there is a hamiltonian cycle in  $\text{Cay}(G/N; S)$ , and
- $s \equiv t \pmod{N}$  for some  $s, t \in S \cup S^{-1}$  with  $s \neq t$ .

Then there is a hamiltonian cycle in  $\text{Cay}(G; S)$ .

**Lemma 2.10** ([2, Lem. 1 on p. 24]). Let  $P_k \square P_\ell$  be the Cartesian product of a path of length  $k$  with a path of length  $\ell$ . If  $k\ell$  is even, and  $k, \ell \geq 2$ , then  $P_k \square P_\ell$  has a hamiltonian path from any corner vertex  $v$  to any vertex that is at odd distance from  $v$ .

**Corollary 2.11.** Suppose  $N$  is a subgroup of an abelian group  $H$ , and  $\{x, y\} \cup S_0$  is a subset of  $H$  that generates  $H/N$ . Let  $k = |\langle x, N \rangle : N|$  and  $|\langle x, y, N \rangle : \langle x, N \rangle|$ . If  $k\ell$  is even,  $k, \ell \geq 2$ ,  $0 \leq p < k$ ,  $0 \leq q < \ell$ , and  $p+q$  is odd, then  $\text{Cay}(H/N; \{x, y\} \cup S_0)$  has a hamiltonian path  $(s_i)_{i=1}^r$ , such that  $s_1 s_2 \cdots s_r = x^p y^q$ .

**Proof.** If we identify the vertices of  $P_k \square P_\ell$  with  $\{(i, j) \mid 0 \leq i < k, 0 \leq j < \ell\}$  in the natural way, then the map  $(i, j) \mapsto x^i y^j$  is an isomorphism from  $P_k \square P_\ell$  to a subgraph  $X$  of  $\text{Cay}(\langle x, y \rangle; \langle x, y \rangle)$ . So Lemma 2.10 provides a hamiltonian path  $(t_i)_{i=1}^{k\ell-1}$  in  $X$  from  $e$  to  $x^p y^q$ . So  $t_1 t_2 \cdots t_{k\ell-1} = x^p y^q$ .

Let  $L = (u_j)_{j=1}^n$  be a hamiltonian path in  $\text{Cay}(H/\langle x, y, N \rangle)$ , and let

$$(s_i)_{i=1}^r = (L, t_{2i-1}, L^{-1}, t_{2i})_{i=1}^{k\ell/2\#}.$$

From the definition of  $k$  and  $\ell$ , we see that the natural map from  $X$  to  $\text{Cay}(\langle x, y, N \rangle / N; x, y)$  is an isomorphism onto a spanning subgraph. Therefore,  $(s_i)_{i=1}^r$  is a hamiltonian path in  $\text{Cay}(H/N; S)$ . Since  $H$  is abelian, it is easy to see that  $s_1 s_2 \cdots s_r = x^p y^q$ . □ [note A.2]

Given a hamiltonian cycle  $C_0$  in  $\text{Cay}(\overline{G}; S)$ , the following result often provides a second hamiltonian cycle  $C_1$ , such that the voltage of at least one of these two cycles generates  $G'$ . (Then the Factor Group Lemma (2.8) provides a hamiltonian cycle in  $\text{Cay}(G; S)$ .)

**Lemma 2.12** (cf. Marušič [13] and Durnberger [4], or see [14, Lem. 3.1]). *Assume:*

- $N$  is an abelian normal subgroup of  $G$ , such that  $G/N$  is abelian,
- $C_0$  is an oriented hamiltonian cycle in  $\text{Cay}(G/N; S)$ ,
- $s, t, u \in S^{\pm 1}$  and  $h \in G$ ,
- $C_0$  contains:
  - the oriented path  $[\overline{hs^{-1}u^{-1}}](s, t, s^{-1})$ , and
  - either the oriented edge  $[\overline{h}](t)$  or the oriented edge  $[\overline{ht}](t^{-1})$ .

Then there is a hamiltonian cycle  $C_1$  in  $\text{Cay}(G/N; S)$ , such that

$$\left( (\Pi C_0)^{-1} (\Pi C_1) \right)^h = \begin{cases} [u, t^{-1}][s, t^{-1}]^u & \text{if } C_0 \text{ contains } [\overline{h}](t), \\ [t^{-1}, u][s, t^{-1}]^u & \text{if } C_0 \text{ contains } [\overline{ht}](t^{-1}). \end{cases}$$

[note A.3]

Furthermore,  $C_0$  and  $C_1$  have exactly the same oriented edges, except for some of the edges in the subgraph induced by  $\{\overline{h}, hu^{-1}, hs^{-1}u^{-1}, \overline{ht}, ht u^{-1}, hts^{-1}u^{-1}\}$ .

**Lemma 2.13** ([4, Lem. 2.8]). *Assume*

- $S$  is an irredundant generating set of  $G$ ,
- $s, t \in S$ , with  $s \neq t$ ,
- $s$  commutes with  $t$ ,
- $\langle S \setminus \{s\} \rangle \triangleleft G$ , and
- there is a hamiltonian cycle in  $\text{Cay}(\langle S \setminus \{s\} \rangle; S \setminus \{s\})$ .

Then there is a hamiltonian cycle in  $\text{Cay}(G; S)$ .

We do not need the general theory of nilpotent groups, but we will make use of the following two facts. (The first is essentially the definition of a nilpotent group, which can be found in any graduate-level textbook on group theory.)

**Lemma 2.14** ([15, (iii) on p. 175 and Prop. VI.1.h on page 176]).

1. Every abelian group is nilpotent.
2. If  $G/Z(G)$  is nilpotent, then  $G$  is nilpotent.

Therefore, if  $G' \subseteq Z(G)$  (in other words, if  $G/Z(G)$  is abelian), then  $G$  is nilpotent.

**Theorem 2.15** ([8]). *If  $G$  is a nontrivial, nilpotent, finite group, and the commutator subgroup of  $G$  is cyclic, then every connected Cayley graph on  $G$  has a hamiltonian cycle.*

The following observation is well known (and easy to prove).

**Lemma 2.16** ([12, Lem. 2.27]). *Let  $S$  generate a finite group  $G$  and let  $s \in S$ , such that  $\langle s \rangle \triangleleft G$ . If*

- *$\text{Cay}(G/\langle s \rangle; S)$  has a hamiltonian cycle, and*
- *either*
  1.  *$s \in Z(G)$ , or*
  2.  *$Z(G) \cap \langle s \rangle = \{e\}$ , or*
  3.  *$|s|$  is prime,*

*then  $\text{Cay}(G; S)$  has a hamiltonian cycle.*

**Corollary 2.17.** *Suppose*

- *$G'$  is cyclic of order  $pq$ , where  $p$  and  $q$  are distinct primes,*
- *$S$  is an irredundant generating set of  $G$ , and*
- *some nontrivial element  $s$  of  $S$  is in  $G'$ .*

*Then  $\text{Cay}(G; S)$  has a hamiltonian cycle.*

**Proof.** We may assume  $G' = \mathbb{Z}_p \times \mathbb{Z}_q$ . Since every subgroup of a cyclic, normal subgroup is also normal, we know that  $\langle s \rangle \triangleleft G$ . Also, there are hamiltonian cycles in  $\text{Cay}(G/\mathbb{Z}_p; S)$ ,  $\text{Cay}(G/\mathbb{Z}_q; S)$ , and  $\text{Cay}(G/G'; S)$  (by Theorem 1.1 and the elementary fact that Cayley graphs on abelian groups have hamiltonian cycles). Hence, we may assume  $\langle s \rangle = G'$  and  $G' \cap Z(G) = \mathbb{Z}_q$  (perhaps after interchanging  $p$  and  $q$ ), for otherwise Lemma 2.16 applies.

Let  $\widehat{G} = G/\mathbb{Z}_p$ . We may assume  $|\widehat{G}| \neq 27$ , for otherwise  $|G| = 27p$  so Theorem 2.3 applies. Then, since  $\widehat{G}$  is nilpotent (see Lemma 2.14) and its commutator subgroup is  $\mathbb{Z}_q$ , the proof in [11, §4] implies there is a hamiltonian cycle  $(t_i)_{i=1}^n$  in  $\text{Cay}(\widehat{G}/\widehat{G}'; S')$  whose voltage generates  $\widehat{G}'$ . Then, since  $\mathbb{Z}_p \cap Z(G) = \{e\}$ , the proof of Lemma 2.16(2) in [12, Lem. 2.27(2)] tells us that  $(t_i, s^{p-1})_{i=1}^n$  is a hamiltonian cycle in  $\text{Cay}(G/\mathbb{Z}_q; S)$ .

Note that, since  $\widehat{G}$  is a nilpotent group whose commutator subgroup is in the center and has prime order  $q$ , the order of  $|\widehat{G}/\widehat{G}'|$  must be a multiple of  $q$ ; that is,  $n$  is a multiple of  $q$  (cf. Lemma 3.6 below). Calculating modulo  $\mathbb{Z}_p$ , we have

$$\begin{aligned} \Pi(t_i, s^{p-1})_{i=1}^n &\equiv s^{(p-1)n} \Pi(t_i)_{i=1}^n & (\widehat{s} \in \widehat{G}' = \widehat{\mathbb{Z}}_q \subseteq Z(\widehat{G})) \\ &\equiv \Pi(t_i)_{i=1}^n & (n \text{ is a multiple of } q) \\ &\neq e & (\Pi(t_i)_{i=1}^n \text{ generates } \widehat{G}'). \end{aligned}$$

Therefore  $\Pi(t_i, s^{p-1})_{i=1}^n$  generates  $\mathbb{Z}_q$ . So the Factor Group Lemma (2.8) tells us that  $((t_i, s^{p-1})_{i=1}^n)^q$  is a hamiltonian cycle in  $\text{Cay}(G; S)$ .  $\square$

### 3 Assumptions, group theory, and connected sums

**Assumptions 3.1.** The remainder of this paper provides a proof of Theorem 1.3, so

- $p$  is an odd prime,
- $G$  is a finite group whose commutator subgroup has order  $2p$ , and
- $S$  is an irredundant generating set of  $G$ .

We wish to show that the Cayley graph  $\text{Cay}(G; S)$  has a hamiltonian cycle.

### 3A Basic group theory

**Assumption 3.2.** Because of Corollary 2.17, we may assume  $S \cap G' = \emptyset$ .

**Notation 3.3.** The assumption that the commutator subgroup has order  $2p$  implies that  $G'$  is cyclic (cf. [14, §2E, proof of Cor. 1.4]), so we may write

$$G' = \mathbb{Z}_2 \times \mathbb{Z}_p.$$

From Theorem 2.15, we may assume  $G$  is not nilpotent, so  $G' \not\subseteq Z(G)$  (see Lemma 2.14). This implies  $\mathbb{Z}_p \cap Z(G) = \{e\}$ . Hence there exists  $a \in S$ , such that [note A.7]

$$a \text{ does not centralize } \mathbb{Z}_p. \quad (3.3A)$$

Then there exists  $b \in S$ , such that

$$\mathbb{Z}_p \subseteq \langle [a, b] \rangle. \quad (3.3B) \text{ [note A.8]}$$

The assumptions (3.3A) and (3.3B) are the basis of most of the arguments in the later sections of the paper.

For ease of reference, we now collect a few well-known facts from group theory (specialized to our setting).

**Lemma 3.4.** *If  $S_0 \subseteq G$ , such that  $\langle S_0, \mathbb{Z}_2 \rangle = G$ , then  $\langle S_0 \rangle = G$ .*

**Proof.** Since  $\mathbb{Z}_2 \subseteq Z(G)$ , we have

$$\langle S_0 \rangle' = \langle S_0, Z(G) \rangle' \supseteq \langle S_0, \mathbb{Z}_2 \rangle' = G'.$$

Therefore

$$\langle S_0 \rangle = \langle S_0, \langle S_0 \rangle' \rangle = \langle S_0, G' \rangle \supseteq \langle S_0, \mathbb{Z}_2 \rangle = G. \quad \square$$

**Corollary 3.5.** *Suppose  $S_0$  is a proper subset of  $S$ , such that  $\mathbb{Z}_p \subseteq \langle S_0 \rangle$ . (In particular, this will be the case if  $\{a, b\} \subseteq S_0$ .) Then  $\langle \overline{S_0} \rangle \neq \overline{G}$ .*

**Proof.** Suppose  $\langle \overline{S_0} \rangle = \overline{G}$ . This means  $\langle S_0, G' \rangle = G$ . Since  $G' = \mathbb{Z}_2 \times \mathbb{Z}_p$  and  $\mathbb{Z}_p \subseteq \langle S_0 \rangle$ , this implies  $\langle S_0, \mathbb{Z}_2 \rangle = G$ . So Lemma 3.4 tells us that  $\langle S_0 \rangle = G$ . This contradicts the fact that the generating set  $S$  is irredundant. □

**Lemma 3.6.** *Let  $H$  be a group. If  $x, y, z \in H$ , and  $y$  centralizes  $H'$ , then  $[xy, z] = [x, z][y, z]$ . [note A.9] Therefore  $[y^k, z] = [y, z]^k$  for all  $k \in \mathbb{Z}$ .*

**Corollary 3.7.** *If  $x, y \in G$ , such that  $y$  centralizes  $G'$ , and  $\mathbb{Z}_p \subseteq \langle [x, y] \rangle$ , then  $|y|$  is divisible by  $p$ . [note A.10]*

**Corollary 3.8.** *Let  $S_0 \subseteq G$ , such that  $\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$ . If  $g \in G$ , such that  $\mathbb{Z}_2 \subseteq \langle g, S_0 \rangle'$ , then  $|\langle \overline{g}, \overline{S_0} \rangle : \langle \overline{S_0} \rangle|$  is even. [note A.11]*

In particular, if  $\mathbb{Z}_2 \subseteq \langle [g, h] \rangle$ , then, by taking  $S_0 = \{h\}$ , we see that  $|\langle \overline{g}, \overline{h} \rangle : \langle \overline{h} \rangle|$  is even, so  $|\overline{g}|$  is even (and, similarly,  $|\overline{h}|$  must also be even).

**Corollary 3.9.**  $|\overline{G}|$  is divisible by 4.

### 3B Connected sums

**Definition 3.10** ([8, Defn. 5.1]). Assume  $C_1$  and  $C_2$  are two vertex-disjoint oriented cycles in  $\text{Cay}(\overline{G}; S)$ , and let  $g \in G$ , and  $s, t \in S \cup S^{-1}$ . If

- $C_1$  contains the oriented edge  $[\overline{g}](t)$ , and

- $C_2$  contains the oriented edge  $[\overline{gst}](t^{-1})$ ,

then we use  $C_1 \#_t^s C_2$  to denote the oriented cycle obtained from  $C_1 \cup C_2$  by

[note A.12]

- removing the oriented edges  $[\overline{g}](t)$  and  $[\overline{gst}](t^{-1})$ , and
- inserting the oriented edges  $[\overline{g}](s)$  and  $[\overline{gst}](s^{-1})$ .

This is called the *connected sum* of  $C_1$  and  $C_2$ .

If  $[g](t)$  is any oriented edge of an oriented cycle  $C$ , and  $s \in S$ , such that  $sC$  is vertex disjoint from  $C$ , then we can form the connected sum  $C \#_t^s -sC$ . This construction can be iterated:

**Definition 3.11.** Suppose

- $[g_1](t_1), \dots, [g_k](t_k)$  are oriented edges of an oriented cycle  $C$  in  $\text{Cay}(\overline{G}; S)$ , such that  $g_i \neq g_{i+1}$  for all  $i$ , and
- $s_1, s_2, \dots, s_k \in S \cup S^{-1}$ , such that the cycles  $C, s_1 C, s_2 s_1 C, \dots, s_k s_{k-1} \dots s_1 C$  are pairwise vertex-disjoint.

Then we can form the connected sum

$$C \#_{t_1}^{s_1} -s_1 C \#_{t_2}^{s_2} s_2 s_1 C \#_{t_3}^{s_3} \dots \#_{t_k}^{s_k} \pm s_k s_{k-1} \dots s_1 C.$$

We call this a *connected sum of signed translates* of  $C$ .

**Lemma 3.12** (cf. [8, Lem. 5.2]). If  $C_1, C_2, g, s$ , and  $t$  are as in Definition 3.10, then

$$\Pi(C_1 \#_t^s C_2) = \Pi C_1 \cdot {}^g[s^{-1}, t^{-1}] \cdot \Pi C_2.$$

**Proof.** We may assume  $g = t^{-1}$  (or, in other words,  $gt = e$ ), after translating the cycles by  $(gt)^{-1}$  (cf. Lemma 2.6(2)). Write  $C_1 = (s_i)_{i=1}^m$  and  $C_2 = [st^{-1}](t_j)_{j=1}^n$ , so

$$(C_1 \#_t^s C_2) = ((s_i)_{i=1}^{m-1}, s, (t_j)_{j=1}^{n-1}, s^{-1}).$$

By assumption,  $C_1$  contains the edge  $\overline{t^{-1}} \rightarrow \overline{e}$  and  $C_2$  contains the edge  $\overline{s} \rightarrow \overline{st^{-1}}$ , so  $s_m = t$  and  $t_n = t^{-1}$ . Therefore

$$\begin{aligned} \Pi(C_1 \#_t^s C_2) &= \prod_{i=1}^{m-1} (s_i) \cdot s \cdot \prod_{j=1}^{n-1} (t_j) \cdot s^{-1} \\ &= \prod_{i=1}^m (s_i) \cdot t^{-1} s \cdot \prod_{j=1}^n (t_j) \cdot ts^{-1} \\ &= \Pi C_1 \cdot t^{-1} s \cdot (\Pi C_2)^{st^{-1}} \cdot ts^{-1} \\ &= \Pi C_1 \cdot t^{-1} st s^{-1} \cdot \Pi C_2 \\ &= \Pi C_1 \cdot {}^{t^{-1}}[s^{-1}, t^{-1}] \cdot \Pi C_2 \\ &= \Pi C_1 \cdot {}^g[s^{-1}, t^{-1}] \cdot \Pi C_2. \end{aligned}$$

□

**Corollary 3.13.** Assume that  $C_1, C_2, g, s$ , and  $t$  are as in Definition 3.10. If  $C_0$  is another oriented cycle that is vertex-disjoint from  $C_2$  and contains the oriented edge  $\overline{g}(t)$ , then

$$(\Pi(C_0 \#_t^s C_2)) (\Pi(C_1 \#_t^s C_2))^{-1} = (\Pi C_0) (\Pi C_1)^{-1}.$$

**Corollary 3.14** ([8, Lem. 5.2]). If  $C_1, C_2, g, s$ , and  $t$  are as in Definition 3.10, then

$$\Pi(C_1 \#_t^s C_2) \equiv \Pi C_1 \cdot \Pi C_2 \cdot [s, t] \pmod{\mathbb{Z}_p}.$$

The following result describes a fairly common situation in which the connected sum provides hamiltonian cycles in  $\text{Cay}(G; S)$ :

**Lemma 3.15.** *Let  $S_0$  be a nonempty subset of  $S$ ,  $g \in G$ ,  $c \in S \setminus S_0$ , and  $s, t \in S \setminus \{c\}$ . Assume  $C_0$  and  $C_1$  are oriented hamiltonian cycles in  $\text{Cay}(\langle \overline{S_0} \rangle; S_0)$ , such that*

- $(\Pi C_0)^{-1}(\Pi C_1)$  is a nontrivial element of  $\mathbb{Z}_p$ ,
- $C_0$  and  $C_1$  both contain the oriented edge  $[\overline{g}](s)$ ,
- for every  $x \in S_0$ ,  $C_0$  contains at least two edges that are labelled either  $x$  or  $x^{-1}$ ,
- $\mathbb{Z}_2 \subseteq \langle [c, t] \rangle$ , and
- either  $|\overline{G} : \langle \overline{S_0} \rangle| > 2$  or  $s = t$ .

If either

1. there exists  $u \in S \setminus \{c\}$ , such that  $\mathbb{Z}_2 \not\subseteq \langle [u, c] \rangle$ , or
2.  $|\overline{G} : \langle \overline{S_0}, \overline{t} \rangle|$  is even,

then there is a hamiltonian cycle  $C$  in  $\text{Cay}(\overline{G}; S)$ , such that  $\langle \Pi C \rangle = G'$ , so the Factor Group Lemma (2.8) yields a hamiltonian cycle in  $\text{Cay}(G; S)$ .

**Proof.** Let  $r = |\overline{G} : \langle \overline{S_0} \rangle|$ . We have  $\mathbb{Z}_p \subseteq \langle (\Pi C_0)^{-1}(\Pi C_1) \rangle \subseteq \langle S_0 \rangle$ , so Corollary 3.5 implies  $r \neq 1$ .

Suppose  $r = 2$ . By assumption, this implies  $s = t$ , which means that  $C_0$  and  $C_1$  both contain the oriented edge  $[\overline{g}](t)$ . Then the translate  $cC_0$  contains the oriented edge  $[\overline{gc}](t)$ . The connected sums  $C = C_0 \#_t^c - cC_0$  and  $C' = C_1 \#_t^c - cC_0$  are hamiltonian cycles in  $\text{Cay}(\overline{G}; S)$ . From Corollary 3.14, we have

$$\Pi C \equiv \Pi C_0 \cdot \Pi C_0 \cdot [c, t] \equiv [c, t] \not\equiv 0 \pmod{\mathbb{Z}_p},$$

so  $\Pi C$  projects nontrivially to  $\mathbb{Z}_2$ . Corollary 3.13 says  $(\Pi C)(\Pi C')^{-1} = (\Pi C_0)(\Pi C_1)^{-1}$ , which generates  $\mathbb{Z}_p$  (because it is conjugate to the inverse of  $(\Pi C_0)^{-1}(\Pi C_1)$ , which is assumed to be a nontrivial element of  $\mathbb{Z}_p$ ). Therefore, we see that either  $\Pi C$  or  $\Pi C'$  generates  $G'$ , as desired. So we may assume henceforth that  $r > 2$ .

We now show that we may assume  $t \in S_0$ . To this end, suppose it is not the case that  $t \in S_0$ . Let  $n = |\langle \overline{S_0}, \overline{t} \rangle : \langle \overline{S_0} \rangle|$ . Then, by choosing a sequence  $\{[g_i](s_i)\}_{i=1}^{n-1}$  of oriented edges of  $C_0$ , we can form a connected sum  $C'_0$  of signed translates of  $C_0$ :

$$C'_0 = C_0 \#_{s_1}^t - tC_0 \#_{s_2}^t \cdots \#_{s_{n-1}}^t \pm t^{n-1}C_0.$$

This is a hamiltonian cycle in  $\text{Cay}(\langle \overline{S_0}, \overline{t} \rangle; S_0 \cup \{t\})$ . We may assume  $s_1 = s$ . Then another hamiltonian cycle  $C'_1$  can be constructed by replacing the leftmost occurrence of  $C_0$  with  $C_1$ , and Lemma 3.12 tells us that  $(\Pi C'_0)(\Pi C'_1)^{-1} = (\Pi C_0)(\Pi C_1)^{-1}$ , which is a nontrivial element of  $\mathbb{Z}_p$  (and  $(\Pi C_0)^{-1}(\Pi C_1)$  is conjugate to the inverse of this). From the definition of connected sum, it is obvious that  $C'_0$  contains at least two edges labelled  $t^{\pm 1}$ . So the hamiltonian cycles  $C'_0$  and  $C'_1$  satisfy the hypotheses of the lemma with  $S_0 \cup \{t\}$  in the role of  $S_0$  and with  $t$  in the role of  $s$ . [note A.13]

*Case 1. Assume there exists  $u \in S \setminus \{c\}$ , such that  $\mathbb{Z}_2 \not\subseteq \langle [u, c] \rangle$ .*

*Subcase 1.1. Assume  $u \in S_0$ . Fix a hamiltonian path  $(s_i)_{i=1}^n$  in  $\text{Cay}(\overline{G} / \langle \overline{S_0} \rangle; S \setminus S_0)$  with  $s_1 = c$ , and let  $\pi_i = \prod_{j=1}^i s_j$ . Any connected sum  $C_0 \#_{t_1}^{s_1} (-\pi_1 C_0) \#_{t_2}^{s_2} \cdots \#_{t_n}^{s_n} (\pm \pi_n C_0)$  is a hamiltonian cycle  $C$  in  $\text{Cay}(\overline{G}; S)$ .*

Since  $[t, c]$  and  $[u, c]$  do not have the same projection to  $\mathbb{Z}_2$ , the voltages of  $C_0 \#_t^c - \pi_1 C_0$  and  $C_0 \#_u^c - \pi_1 C_0$  do not have the same projection to  $\mathbb{Z}_2$ . Therefore, by choosing  $t_1$  to be the



appropriate element of  $\{t, u\}$ , we may assume the projection of  $\Pi C$  to  $\mathbb{Z}_2$  is nontrivial (see [note A.14] Corollary 3.14). Note also that if  $|\overline{G} : \langle \overline{S_0} \rangle| = 2$ , then we must have  $t_1 = t$ .

We may assume that  $t_n = s$ , and that the connected sum  $(-1)^{n-1} \pi_{n-1} C_0 \#_{s^n}^{s_n} (-1)^n \pi_n C_0$  [note A.15] is relative to the oriented edge  $[\pi_n g](s)$  of  $\pi_n C_0$  that is also in  $\pi_n C_1$ . Therefore, another hamiltonian cycle  $C'$  can be constructed by replacing  $\pi_n C_0$  with  $\pi_n C_1$  in the connected sum. Then Lemma 3.12 (together with Lemma 2.6(2)) implies that  $(\Pi C)^{-1}(\Pi C')$  is conjugate to  $(\Pi C_0)^{-1}(\Pi C_1)$ , which is a generator of  $\mathbb{Z}_p$ . Therefore, either  $\Pi C$  or  $\Pi C'$  generates  $G'$ , as desired.

*Subcase 1.2. Assume  $u \notin S_0$ .* Let  $S_u = \{u\} \cup S_0$ , let  $n = |\langle \overline{S_u} \rangle : \langle \overline{S_0} \rangle| - 1$ , let  $(s_i)_{i=1}^n$  be a hamiltonian path in  $\text{Cay}(\overline{G}/\langle \overline{S_u} \rangle; S \setminus S_u)$  with  $s_1 = c$ , and let  $\pi_i = \prod_{j=1}^i s_j$ . (Since  $S \setminus S_0$  is an irredundant generating set for  $\overline{G}/\langle \overline{S_0} \rangle$ , we have  $m, n \geq 1$ .) Any connected sum

$$C_u = C_0 \#_{t_1}^u - u C_0 \#_{t_2}^u \cdots \#_{t_n}^u \pm u^n C_0$$

is a hamiltonian cycle in  $\text{Cay}(\langle \overline{S_u} \rangle; S_u)$ , so any connected sum

$$C = C_u \#_{t_1'}^{s_1} - \pi_1 C_u \#_{t_2'}^{s_2} \cdots \#_{t_m'}^{s_m} \pm \pi_m C_u$$

is a hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$ .

Since  $t \in S_0$ , we know that  $C_0$  contains more than one edge labeled  $t^{\pm 1}$ , so  $-u C_0$  has an edge labeled  $t^{\pm 1}$  that was not removed in the construction of the connected sum  $C_0 \#_{t_1}^u - \pi_1 C_0$ . Furthermore, the definition of the connected sum implies that  $C_0 \#_{t_1}^u - \pi_1 C_0$  also contains an edge labeled  $u$ . Therefore, we may form connected sums

$$C_u \#_{t^{\pm 1}}^c - \pi_1 C_u \text{ and } C_u \#_u^c - \pi_1 C_u$$

without removing any of the edges of  $C_u$ . Since  $[c, t]$  and  $[c, u]$  do not have the same projection to  $\mathbb{Z}_2$ , the voltages of these two connected sums do not have the same projection to  $\mathbb{Z}_2$  (see Corollary 3.14). Therefore, by choosing  $t_1'$  to be the appropriate element of  $\{t^{\pm 1}, u\}$ , we may assume the projection of  $\Pi C$  to  $\mathbb{Z}_2$  is nontrivial.

We have

$$C = C_u \#_{t_1'}^{s_1} - \pi_1 C_u \#_{t_2'}^{s_2} \cdots \#_{t_{m-1}'}^{s_{m-1}} \pm \pi_{m-1} C_u \#_{t_m'}^{s_m} (\pm \pi_m C_0 \#_{t_1}^u \pm \pi_m u C_0 \#_{t_2}^u \cdots \#_{t_n}^u \pm \pi_m u^n C_0),$$

so the proof can be completed almost exactly as in the final paragraph of Subcase 1.1 (by constructing another connected sum in which  $\pi_m u^n C_0$  is replaced with  $\pi_m u^n C_1$ ).

*Case 2. Assume  $[u, c]$  projects nontrivially to  $\mathbb{Z}_2$ , for every  $u \in S \setminus \{c\}$ .* In particular,  $[d, c]$  projects nontrivially to  $\mathbb{Z}_2$ , for every  $d \in S \setminus (S_0 \cup \{c\})$ . Since we may assume that Case 1 does not apply with  $d$  in the place of  $c$ , we conclude that we may assume

$$[u, d] \text{ projects nontrivially to } \mathbb{Z}_2, \text{ for all } d \in S \setminus S_0 \text{ and } u \in S \setminus \{d\}. \quad (3.15A)$$

Choose a hamiltonian path  $(s_i)_{i=1}^n$  in  $\text{Cay}(\overline{G}/\langle \overline{S_0} \rangle; S \setminus S_0)$ . Any connected sum

$$C = C_0 \#_{t_1}^{s_1} - \pi_1 C_0 \#_{t_2}^{s_2} \cdots \#_{t_n}^{s_n} \pm \pi_n C_0$$

is a hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$ . Calculating modulo  $\mathbb{Z}_p$ , and letting  $z$  be the nontrivial element of  $\mathbb{Z}_2$ , we have

$$\begin{aligned} \Pi C &\equiv \Pi C_0 \cdot [s_1, t_1] \cdot \Pi(-\pi_1 C_0) \cdots [s_n, t_n] \cdot \Pi(\pm \pi_n C_0) && \text{(Corollary 3.14)} \\ &\equiv \Pi C_0 \cdot z \cdot \Pi C_0 \cdots z \cdot \Pi C_0 && \text{(Lemma 2.6(2) and (3.15A))} \\ &= (\Pi C_0)^{n+1} \cdot z^n \\ &\equiv z && (n \text{ is odd}). \end{aligned}$$

The proof is now completed exactly as in the final paragraph of Subcase 1.1.  $\square$

**Corollary 3.16.** *Let  $S_0 \subseteq S$ ,  $g \in G$ , and  $s \in S_0$ . Assume  $C_0$  and  $C_1$  are oriented hamiltonian cycles in  $\text{Cay}(\langle \overline{S_0}; S_0)$ , such that*

- $(\Pi C_0)^{-1}(\Pi C_1)$  is a nontrivial element of  $\mathbb{Z}_p$ ,
- $C_0$  and  $C_1$  both contain the oriented edge  $[\overline{g}](s)$ ,
- for every  $x \in S_0$ ,  $C_0$  contains at least two edges that are labelled either  $x$  or  $x^{-1}$ , and
- $\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$ .

*Then there is a hamiltonian cycle  $C$  in  $\text{Cay}(\overline{G}; S)$ , such that  $\langle \Pi C \rangle = G'$ , so the Factor Group Lemma (2.8) yields a hamiltonian cycle in  $\text{Cay}(G; S)$ .*

**Proof.** We may assume  $[c, t] \in \mathbb{Z}_p$ , for all  $c \in S$  and  $t \in S_0$ . (Otherwise, we see from Corollary 3.8 that Lemma 3.15(2) applies.) Choose  $c, d \in S$ , such that  $[c, d] \notin \mathbb{Z}_p$ , let  $S_0^+ = S_0 \cup \{d\}$ , and let  $r = |\langle S_0^+ \rangle : \langle S_0 \rangle|$ . Any connected sum of the following form is a hamiltonian cycle in  $\text{Cay}(\langle S_0^+ \rangle; S_0^+)$ :

$$C = C_0 \#_{s_1}^d - d C_0 \#_{s_2}^d \cdots \#_{s_{r-1}}^d \pm d^{r-1} C_0.$$

We may assume  $s_1 = s$ , and that the connected sum  $C_0 \#_{s_1}^d - d C_0$  is formed by using the oriented edge  $[\overline{g}](s)$  that is also in  $C_1$ . Therefore, a second hamiltonian cycle  $C'$  can be constructed by replacing the leftmost occurrence of  $C_0$  with  $C_1$ . Then Corollary 3.8 implies that Lemma 3.15(2) applies (with  $S_0^+$ ,  $d$ ,  $d$ ,  $C$ , and  $C'$  in the roles of  $S_0$ ,  $s$ ,  $t$ ,  $C_0$ , and  $C_1$ , respectively).  $\square$

## 4 Case with $\overline{s} = \overline{t}$

**Case 4.1.** *Assume there exist  $s, t \in S \cup S^{-1}$  with  $\overline{s} = \overline{t}$  and  $s \neq t$ .*

**Proof.** Write  $t = s\gamma$  with  $\gamma \in G'$ . We may assume  $\langle \gamma \rangle = G'$ , for otherwise  $|\gamma|$  is prime, so Corollary 2.9 applies with  $N = \langle \gamma \rangle$ . Note that the irredundance of  $S$  implies  $\langle S \setminus \{s\} \rangle$  and  $\langle S \setminus \{t\} \rangle$  do not contain  $\mathbb{Z}_p$ . This implies that every element of  $S \setminus \{s, t\}$  centralizes  $\mathbb{Z}_p$ . [note A.17]  
So  $s$  and  $t$  do not centralize  $\mathbb{Z}_p$ . [note A.18]

Let  $m = |\overline{t}|$  and  $n = |\overline{G}|/m$ .

**Subcase 4.1.1.** *Assume  $|\overline{t}| > 2$ . Since  $\overline{G}$  is abelian, it is easy to find a hamiltonian cycle  $C = (t_i)_{i=1}^{mn}$  in  $\text{Cay}(\overline{G}; S \setminus \{s\})$ , such that  $t_1 = t_2 = \cdots = t_{m-1} = t$ . Since  $\langle \Pi C \rangle \subseteq \langle S \setminus \{s\} \rangle$ , [note A.19] and  $\mathbb{Z}_p \not\subseteq \langle S \setminus \{s\} \rangle$ , we must have  $\Pi C \in \mathbb{Z}_2$ .*

For each subset  $I$  of  $\{1, \dots, m-1\}$ , we define  $C_I$  to be the hamiltonian cycle constructed from  $C$  by changing  $t_i$  to  $s$  for all  $i \in I$ . The proof is completed by noting that  $I$  may be chosen such that  $\Pi C_I$  generates  $G'$ , so the Factor Group Lemma (2.8) applies:

- If  $\Pi C = e$ , let  $I = \{1\}$ .
- If  $\Pi C$  is the nontrivial element of  $\mathbb{Z}_2$ , and  $t$  does not invert  $\mathbb{Z}_p$ , then we may let  $I = \{1, 2\}$ .
- If  $\Pi C$  is the nontrivial element of  $\mathbb{Z}_2$ , and  $t$  inverts  $\mathbb{Z}_p$ , then  $|\overline{t}|$  is even, so we must have  $|\overline{t}| \geq 4$ . We may let  $I = \{1, 3\}$ .

**Subcase 4.1.2.** *Assume  $|\overline{t}| = 2$ . (Since  $t$  does not centralize  $\mathbb{Z}_p$ , this implies that  $t$  inverts  $\mathbb{Z}_p$ .) Choose a hamiltonian cycle  $(s_i)_{i=1}^n$  in  $\text{Cay}(\overline{G}/\langle \overline{t} \rangle; S \setminus \{s, t\})$ , and let*

$$C_0 = (t, s_i)_{i=1}^n = (t_j)_{j=1}^{2n}.$$

[note A.20] Since  $n = |\bar{G}|/2$  is even (see Corollary 3.9) and  $S \setminus \{s\}$  is an irredundant generating set of  $\bar{G}$ , it is easy to see that  $C_0$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S \setminus \{s\})$ . Note that  $t_i = t$  [note A.21] whenever  $i$  is odd, and that  $\Pi C_0 \in \mathbb{Z}_2$  (because  $\mathbb{Z}_p \not\subseteq \langle S \setminus \{s\} \rangle$ ). [note A.22]

We may assume  $n \geq 6$  (for otherwise  $|G| = 4np \leq 20p$ , so Theorem 2.3 applies). We construct a hamiltonian cycle  $C_1$  from  $C_0$ :

- If  $\Pi C_0 = e$ , construct  $C_1$  by changing  $t_1$  to  $s$ .
- If  $\Pi C_0 \neq e$ , construct  $C_1$  by changing both  $t_1$  and  $t_5$  to  $s$ .

In each case,  $\Pi C_1$  generates  $G'$ . (To see this in the second case, note that  $t_2 t_3 t_4 t_5 = s_1 t s_2 t$  centralizes  $G'$ , because  $t$  inverts  $G'$ , and each  $s_i$  centralizes  $G'$ .) Therefore, the Factor Group Lemma (2.8) applies.  $\square$

## 5 Cases with $|\bar{a}| > 2$ and $\bar{b} \notin \langle \bar{a} \rangle$

Recall that the elements  $a$  and  $b$  of  $S$  satisfy (3.3A) and (3.3B).

**Case 5.1.** Assume  $|\bar{a}| > 2$ ,  $\bar{b} \notin \langle \bar{a} \rangle$ , and there exists  $c \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ . (It may be the case that  $b = c$ .)

**Proof.** Let  $m = |\bar{a}|$  and  $n = |\bar{G} : \langle \bar{a} \rangle|$ . Since  $\bar{b}, \bar{c} \notin \langle \bar{a} \rangle$  (and  $\bar{G}/\langle \bar{a} \rangle$  is abelian), it is easy to find a hamiltonian cycle  $(s_i)_{i=1}^n$  in  $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; S \setminus \{a\})$ , such that  $s_n \in \{c^{\pm 1}\}$ , and  $s_k = b$  for some  $k < n$ . Since  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ , we know  $m$  and  $n$  are both even (see Corollary 3.8). Since  $n$  is even, we have the following (well-known) hamiltonian cycle  $C_0$  in  $\text{Cay}(\bar{G}; \bar{S})$ :

$$C_0 = (a, (a^{m-2}, s_{2i-1}, a^{-(m-2)}, s_{2i})_{i=1}^{n/2}, a^{-1}, (s_{n-j}^{-1})_{j=1}^{n-1}). \quad (5.1A)$$

[note A.23] Letting  $\hat{G} = G/\mathbb{Z}_p$ , we have  $\hat{G}' = \mathbb{Z}_2$ , so  $\hat{a}^{m-2} \in Z(\hat{G})$  (because  $m$  is even). Therefore

$$a^{m-2} s_{2i-1} a^{-(m-2)} \equiv s_{2i-1} \pmod{\mathbb{Z}_p},$$

so, calculating modulo  $\mathbb{Z}_p$ , we have

$$\Pi C_0 \equiv a \cdot \left( \prod_{i=1}^{n-1} s_j \right) \cdot a^{-1} \cdot \left( \prod_{i=1}^{n-1} s_j \right)^{-1} \equiv a \cdot s_n^{-1} \cdot a^{-1} \cdot s_n = [a^{-1}, s_n] = [a^{-1}, c^{\pm 1}],$$

which is nontrivial (mod  $\mathbb{Z}_p$ ).

Recall that  $s_k = b$ . Let  $g = \prod_{i=1}^{k-1} s_i$  and  $\delta = (-1)^{k+1}$ . Then  $C_0$  contains both the oriented edge  $[\overline{gb}](b^{-1})$  and the oriented path  $[\overline{ga^{-2\delta}}](a^\delta, b, a^{-\delta})$ . So Lemma 2.12 (with  $s = a^\delta$ ,  $t = b$ ,  $u = a^\delta$  and  $h = g$ ) provides a hamiltonian cycle  $C_1$ , such that  $(\Pi C_0)^{-1}(\Pi C_1)$  is conjugate to  $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta}$ . Since  $a$  centralizes  $\mathbb{Z}_2$ , but not  $\mathbb{Z}_p$ , this voltage is a generator of  $\mathbb{Z}_p$ . [note A.25]

Thus, either  $\Pi C_0$  or  $\Pi C_1$  generates  $\mathbb{Z}_2 \times \mathbb{Z}_p = G'$ , so the Factor Group Lemma (2.8) provides a hamiltonian cycle in  $\text{Cay}(G; S)$ .  $\square$

**Case 5.2.** Assume  $|\bar{a}| > 2$ ,  $\bar{b} \notin \langle \bar{a} \rangle$ , and there does not exist  $c \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ .

**Proof.** Choose  $c, d \in S$  with  $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$ . Let

$$m = |\bar{a}|, \quad n = |\langle \bar{S} \setminus \{\bar{d}\} \rangle|/m, \quad \text{and} \quad r = |\bar{G}|/(mn).$$

By assumption, we know  $a \notin \{c, d\}$ . Also, we may assume  $d \neq b$  (after interchanging  $c$  and  $d$  if necessary). Then Corollary 3.5 tells us  $r > 1$ . Furthermore, we see from

[note A.26] Corollary 3.8 that the image of  $c$  in  $\bar{G}/\langle \bar{a} \rangle$  has even order, so  $n$  is even.

**Subcase 5.2.1.** Assume  $n > 2$ . It is not difficult to construct a hamiltonian cycle  $(s_i)_{i=1}^n$  in  $\text{Cay}(\langle \bar{S} \setminus \{\bar{d}\} \rangle / \langle \bar{a} \rangle; \bar{S} \setminus \{\bar{a}, \bar{d}\})$ , such that  $s_1 = b$  and  $s_k = c^{\pm 1}$  for some  $k \notin \{1, n\}$ . [note A.27] Then, since  $n$  is even, we may define  $C_0$  as in (5.1A), so  $C_0$  is a hamiltonian cycle in  $\text{Cay}(\langle \bar{S} \setminus \{\bar{d}\} \rangle; S \setminus \{d\})$ .

Let  $g = s_1 s_2 \cdots s_k$ , and note that  $C_0$  contains the oriented edges  $[\bar{e}](a)$  and  $[\bar{g}](c^{\pm 1})$ . Since  $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$ , but  $\mathbb{Z}_2 \not\subseteq \langle [a, d] \rangle$ , we see from Lemma 3.12 that there is a connected sum

$$C = C_0 \#_{t_1}^d - dC_0 \#_{t_2}^d \cdots \#_{t_{r-1}}^d \pm d^{r-1}C_0,$$

with  $t_1 \in \{a, c^{\pm 1}\}$ , such that  $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$ . Note that  $C$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ .

The cycle  $C_0$  contains both  $[\bar{b}](b^{-1})$  and  $[\bar{a}^{-2}](a, b, a^{-1})$ , and neither of these paths contains either the edge  $[\bar{e}](a)$  or the edge  $[\bar{g}](c^{\pm 1})$ . Therefore,  $C$  also contains both of these paths, so Lemma 2.12 (with  $s = a$ ,  $t = b$ ,  $u = a$ , and  $h = e$ ) provides a hamiltonian cycle  $C'$  in  $\text{Cay}(\bar{G}; S)$ , such that  $(\Pi C)^{-1}(\Pi C')$  is a conjugate of  $[b^{-1}, a][a, b^{-1}]^a$ , which is a generator of  $\mathbb{Z}_p$  (since  $a$  centralizes  $\mathbb{Z}_2$ , but not  $\mathbb{Z}_p$ ). Then either  $\Pi C$  or  $\Pi C'$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 5.2.2.** Assume  $n = 2$  and  $r > 2$ . Since  $n = 2$  (and  $\bar{b} \notin \langle \bar{a} \rangle$ ), we have  $\langle \bar{a}, \bar{b}, \bar{d} \rangle = \bar{G}$ , so Corollary 3.5 implies  $S = \{a, b, d\}$ . (Therefore  $b = c$ , which means  $\mathbb{Z}_2 \subseteq \langle [b, d] \rangle$ .) We have the following hamiltonian cycle in  $\text{Cay}(\langle \bar{a}, \bar{b} \rangle; \bar{a}, \bar{b})$ :

$$C_0 = [\bar{e}](a^{m-1}, b, a^{-(m-1)}, b^{-1}).$$

Using the oriented edge  $[\bar{e}](a)$ , we can form the connected sum  $C_0 \#_{t_1}^d - dC_0$ . Then, since  $dC_0$  contains both  $[\bar{d}b](b^{-1})$  and  $[\bar{d}ab](a^{-1})$ , we can extend this to a connected sum

$$C = C_0 \#_{t_1}^d - dC_0 \#_{t_2}^d \cdots \#_{t_{r-1}}^d \pm d^{r-1}C_0,$$

with  $t_2 \in \{a, b\}$ , such that  $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$  (see Corollary 3.14). Since  $C$  contains both  $[\bar{b}](b^{-1})$  and  $[\bar{a}^{-2}](a, b, a^{-1})$ , we may argue as in the last paragraph of Subcase 5.2.1. Namely, Lemma 2.12 (with  $s = a$ ,  $t = b$ ,  $u = a$ , and  $h = e$ ) provides a hamiltonian cycle  $C'$  in  $\text{Cay}(\bar{G}; S)$ , such that  $(\Pi C)^{-1}(\Pi C')$  is a conjugate of  $[b^{-1}, a][a, b^{-1}]^a$ , which is a generator of  $\mathbb{Z}_p$ . Then either  $\Pi C$  or  $\Pi C'$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 5.2.3.** Assume  $n = r = 2$ . As in Subcase 5.2.2, we must have  $S = \{a, b, d\}$  and  $b = c$  (so  $\mathbb{Z}_2 \subseteq \langle [b, d] \rangle$ ).

**Subsubcase 5.2.3.1.** Assume  $m \neq 3$ . Since  $m = |\bar{a}| > 2$  (by an assumption of this case), we have  $m \geq 4$ . We have the following hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ :

$$C_0 = (d, b, a, b^{-1}, d^{-1}, a^{m-2}, d, a^{-(m-3)}, b, a^{m-3}, d^{-1}, a^{-(m-1)}, b^{-1}).$$

[note A.28]

Since  $a$  is central in  $G/\mathbb{Z}_p$  (by an assumption of this case), we know that

$$\Pi C_0 \equiv dbb^{-1}d^{-1}dbd^{-1}b^{-1} = dbd^{-1}b^{-1} = [d^{-1}, b^{-1}] \equiv [d, b] = [d, c] \pmod{\mathbb{Z}_p},$$

so  $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$ .

Note that  $C_0$  contains both  $[\bar{d}ab](b^{-1})$  and  $[\bar{d}a^3](a^{-1}, b, a)$  (because  $m \geq 4$ ), so applying Lemma 2.12 (with  $s = a^{-1}$ ,  $t = b$ ,  $u = a^{-1}$  and  $h = da$ ) yields a hamiltonian cycle  $C_1$  in  $\text{Cay}(G; S)$ , such that  $(\Pi C_0)^{-1}(\Pi C_1)$  is a conjugate of  $[b^{-1}, a^{-1}][a^{-1}, b^{-1}]^{a^{-1}}$ , which is a generator of  $\mathbb{Z}_p$ . Then either  $\Pi C$  or  $\Pi C'$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subsubcase 5.2.3.2.** Assume  $m = 3$  and  $d$  does not centralize  $G'$ . Since the walk  $(a^{-2}, b^{-1}, a^2)$  is a hamiltonian path in  $\text{Cay}(\langle \bar{a}, \bar{b} \rangle; a, b)$ , we have the following hamiltonian

cycle in  $\text{Cay}(\bar{G}; S)$ :

$$C = (a^{-2}, b^{-1}, a^2, d^{-1}, a^{-2}, b, a^2, d).$$

Note that

$$\Pi C = (a^{-2}b^{-1}a^2)d^{-1}(a^{-2}ba^2)d = (b^{a^2})^{-1}d^{-1}(b^{a^2})d = [b^{a^2}, d].$$

[note A.29] Since  $a^2$  does not invert  $G'$ , we know that  $b^{a^2} \not\equiv b^{a^{-2}} \pmod{\mathbb{Z}_2}$ . Therefore, since  $d$  does [note A.30] not centralize  $G'$ , we may assume  $[b^{a^2}, d] \not\equiv e \pmod{\mathbb{Z}_2}$  (by replacing  $a$  with its inverse if necessary). Also, since  $G'$  is central modulo  $\mathbb{Z}_p$ , we have  $[b^{a^2}, d] \equiv [b, d] \not\equiv e \pmod{\mathbb{Z}_p}$ . Therefore,  $\Pi C$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subsubcase 5.2.3.3.** Assume  $m = 3$  and  $d$  centralizes  $G'$ . Suppose  $[b, d] \in \mathbb{Z}_2$ . Let  $\hat{G} = G/\mathbb{Z}_2$  and  $\hat{H} = \langle \hat{a}, \hat{b} \rangle$ . From Theorem 1.1, we know there is a hamiltonian cycle in  $\text{Cay}(\hat{H}; a, b)$ . Deleting an edge labeled  $b^{\pm 1}$  (and passing to the reverse and/or a translate if necessary) yields a hamiltonian path  $L = (t^i)_{i=1}^{2mp-1}$  in  $\text{Cay}(\hat{H}; a, b)$  from  $\hat{e}$  to  $\hat{b}$ . Let

$$C = (L^{-1}, d^{-1}, L, d).$$

Then

$$\Pi C = [\prod_{i=1}^{2mp-1} t_i, d] \in [b\mathbb{Z}_2, d] = \{[b, d]\},$$

because  $\mathbb{Z}_2$  is in the center of  $G$ . Since  $[b, d] \in \mathbb{Z}_2$ , this calculation implies that  $C$  is a closed walk in  $G/\mathbb{Z}_2 = \hat{G}$ . So  $C$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ . The calculation also implies that the Factor Group Lemma (2.8) applies, because  $\langle [b, d] \rangle = \mathbb{Z}_2$ .

We may now assume  $[b, d] \notin \mathbb{Z}_2$ . Therefore, since  $d$  centralizes  $G'$ , and  $p^2 \nmid 12 = |\bar{G}|$ , [note A.31] we see from Lemma 3.6 that  $b$  does not centralize  $G'$ . Also, we may assume  $[a, d] \neq e$ , for otherwise Lemma 2.13 applies with  $s = d$  and  $t = a$ . However, we know  $\mathbb{Z}_2 \not\subseteq \langle [a, d] \rangle$  (by an assumption of this case). Therefore  $\langle [a, d] \rangle = \mathbb{Z}_p$ . So Subsubcase 5.2.3.2 applies after interchanging  $b$  and  $d$ .  $\square$

## 6 Cases with $\bar{b} \in \langle \bar{a} \rangle$

**Case 6.1.** Assume  $\bar{b} \in \langle \bar{a} \rangle$  and  $a$  does not invert  $G'$ .

**Proof.** Let  $m = |\bar{a}|$ . We may assume (perhaps after replacing  $b$  with its inverse) that we may write  $b = a^k \gamma$  with  $1 \leq k \leq m/2$  and  $\gamma \in G'$ . Assume  $k \geq 2$ , for otherwise Case 4.1 applies. This implies  $m - 1 \geq k + 1$  (since  $m = |\bar{a}| \geq 2k \geq k + 2$ ).

**Subcase 6.1.1.** Assume there exists  $c \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ . Let  $n = |\bar{G} : \langle \bar{a} \rangle|$ . Note that Corollary 3.8 implies  $m$  and  $n$  are even, and  $c \notin \langle \bar{a} \rangle$  (so  $c \neq b$ ).

Choose a hamiltonian cycle  $(s_i)_{i=1}^n$  in  $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; S \setminus \{a, b\})$ , such that  $s_n = c$ , and define  $C_0$  as in (5.1A). Then  $\langle \Pi C_0 \rangle$  contains  $\mathbb{Z}_2$  by the same calculation as in Case 5.1.

Since  $m - 1 \geq k + 1$ , we may construct a hamiltonian cycle  $C_1$  in  $\text{Cay}(\bar{G}; S)$  by replacing the path  $(a^{k+1})$  at the start of  $C_0$  with  $(b, a^{-(k-1)}, b)$ . Then

$$(\Pi C_1)(\Pi C_0)^{-1} = ba^{-(k-1)}ba^{-(k+1)} = (a^k \gamma)a^{-(k-1)}(a^k \gamma)a^{-(k+1)} = a^{k+1} \gamma^a \gamma a^{-(k+1)}.$$

This is a generator of  $\mathbb{Z}_p$ , since  $a$  inverts  $\mathbb{Z}_2$ , but not  $\mathbb{Z}_p$ . Hence, either  $\Pi C_0$  or  $\Pi C_1$  generates  $G'$ , so the Factor Group Lemma (2.8) provides a hamiltonian cycle in  $\text{Cay}(G; S)$ .

**Subcase 6.1.2.** Assume there does not exist  $c \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ . Choose  $c, d \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$ . (It is possible that  $b \in \{c, d\}$ , but we know, by the assumption of this subcase, that  $a \notin \{c, d\}$ .) Let  $n = |\langle \bar{a}, \bar{d} \rangle : \langle \bar{a} \rangle|$  and  $r = |\bar{G}|/(mn)$ . From Corollary 3.8 (and the assumption of this subcase), we know  $n$  and  $r$  are even.

We have the following hamiltonian cycle in  $\text{Cay}(\langle \bar{a}, \bar{d} \rangle; a, d)$ :

$$C_0 = ((a, (a^{m-2}, d, a^{-(m-2)}, d)^{n/2\#}, a^{-1}, d^{-(n-1)})). \quad [\text{note A.32}]$$

As in the final paragraph of Subcase 6.1.1, another hamiltonian cycle  $C_1$  can be constructed by replacing the path  $(a^{k+1})$  at the start of  $C_0$  with  $(b, a^{-(k-1)}, b)$ , and the calculation in Subcase 6.1.1 shows that  $(\Pi C_1)(\Pi C_0)^{-1}$  generates  $\mathbb{Z}_p$ . Therefore, since  $[c, d] \notin \mathbb{Z}_p$ , but  $[c, a] \in \mathbb{Z}_p$ , we see that Lemma 3.15(1) applies (with  $S_0 = \{a, b, d\}$ ,  $g = a^{-1}$ ,  $s = t = d$ , and  $u = a$ ).  $\square$

**Case 6.2.** Assume  $\bar{b} \in \langle \bar{a} \rangle$  and  $a$  inverts  $G'$ .

**Proof.** As in Case 6.1, we let  $m = |\bar{a}|$  and write  $b = a^k \gamma$  with  $2 \leq k \leq m/2$  and  $\gamma \in G'$ . We now consider the same five subcases as in [4, pp. 60–62].

**Subcase 6.2.1.** Assume  $2 < k < m/2$  and  $k$  is even. Let  $C_1 = (a^m)$ . The proof in the last paragraph of [4, p. 60] provides a hamiltonian cycle

$$C_0 = (b, a^{-(k-4)}, b, a^{m-2k-2}, b, a^{-1}, b, a^2, b^{-2}, a^{k-3})$$

in  $\text{Cay}(\langle \bar{a} \rangle; a, b)$ , such that  $(\Pi C_0)^{-1}(\Pi C_1)$  is a generator of  $\mathbb{Z}_p$ . Therefore, Corollary 3.16 [note A.33] applies (with  $S_0 = \{a, b\}$ ), because  $C_0$  and  $C_1$  both contain the oriented edge  $[\bar{a}^{-1}](a)$ .

**Subcase 6.2.2.** Assume  $2 < k < m/2$  and  $k$  is odd. Let

$$C_0 = ((b, a, b^{-1}, a)^{(k-1)/2}, b, a^{m-2k+1})$$

and

$$C_1 = ((b, a^{-1}, b^{-1}, a^{-1})^{(k-1)/2}, b^2, a^{m-2k-1}, b).$$

Calculations in [4, p. 61] show that  $(\Pi C_0)^{-1}(\Pi C_1)$  is a generator of  $\mathbb{Z}_p$ . Therefore, Corollary 3.16 applies (with  $S_0 = \{a, b\}$ ), because  $C_0$  and  $C_1$  both contain the oriented edge  $[\bar{e}](b)$ .

**Subcase 6.2.3.** Assume  $k = m/2$  and  $k$  is even. We follow the argument of [11, Subcase iii, p. 97]. Since  $k$  is even, we know  $a^k$  centralizes  $G'$ , so

$$b^2 = (a^k \gamma)^2 = a^{2k} \gamma^2 = a^m \gamma^2 \in \mathbb{Z}_2 \cdot \gamma^2 \not\cong e.$$

Therefore Corollary 2.9 applies (with  $s = b$  and  $t = b^{-1}$ ).

**Subcase 6.2.4.** Assume  $k = m/2$  and  $k$  is odd. Choose  $c \in S$  so that  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ , if such  $c$  exists. Otherwise, choose  $c$  so that there exists  $d \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$ . In either case, Corollary 3.8 implies  $c \in S \setminus \{a, b\}$ , and  $|\langle \bar{a}, \bar{c} \rangle : \langle \bar{a} \rangle|$  is even.

We may assume  $b^2 = e$ , for otherwise Corollary 2.9 applies (with  $s = b$  and  $t = b^{-1}$ ). Therefore, noting that  $a^k$  inverts  $G'$  (since  $k$  is odd), we have

$$e = b^2 = (a^k \gamma)(a^k \gamma) = a^{2k} \cdot \gamma^{-1} \gamma = a^m.$$

**Subsubcase 6.2.4.1.** Assume  $|\bar{G} : \langle \bar{a} \rangle| > 2$ . It suffices to find a hamiltonian cycle  $C_*$  in  $\text{Cay}(\bar{G}; S)$ , such that  $\Pi C_*$  projects nontrivially to  $\mathbb{Z}_2$ , and  $C_*$  contains the paths  $[\bar{a}^{k-3}](a, b, a^{-1})$  and  $[\bar{a}^{k-1}b](b^{-1})$ . For then Lemma 2.12 (with  $s = a$ ,  $t = b$ ,  $u = a$ , and  $h = a^{k-1}$ ) provides a hamiltonian cycle  $C'_*$ , such that  $\langle (\Pi C_*)^{-1}(\Pi C'_*) \rangle = \mathbb{Z}_p$ . Therefore, either  $\Pi C_*$  or  $\Pi C'_*$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

Note that

$$C = (a^{k-2}, b, a^{-(k-2)}, c, a^{k-1}, c^{-1}, b^{-1}, c, a^{-(k-1)}, c^{-1}) \quad [\text{note A.34}]$$

is a cycle through the vertices of  $\text{Cay}(\overline{G}; \{a, b, c\})$  in  $\langle \bar{a} \rangle \cup c\langle \bar{a} \rangle$ . A connected sum of translates of  $C$  yields a hamiltonian cycle  $C_0$  in  $\text{Cay}(\overline{G}; S)$ .

If  $\mathbb{Z}_2 \not\subseteq \langle [a, c] \rangle$ , then the connected sum defining  $C_0$  can be chosen so that  $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$  (see the proof of Lemma 3.15). So we may let  $C_* = C_0$ .

We may now assume  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ . Construct a hamiltonian cycle  $C_1$  in  $\text{Cay}(\overline{G}; S)$  by replacing the rightmost translate of  $C$  in the connected sum with

$$C' = (a^{k-1}, b, a^{-(k-1)}, c, a^{k-1}, b^{-1}, a^{-(k-1)}, c^{-1}).$$

A straightforward calculation shows that  $(\Pi C)^{-1}(\Pi C') \notin \mathbb{Z}_p$ , so we have  $\mathbb{Z}_2 \subseteq \langle \Pi C_i \rangle$  for some  $i \in \{0, 1\}$ . Let  $C_* = C_i$ .

**Assumptions 6.2.4.2.** We may now assume  $|\overline{G} : \langle \bar{a} \rangle| = 2$ , so the irredundance of  $S$  implies  $S = \{a, b, c\}$ . Since  $\bar{b} \in \langle \bar{a} \rangle$ , the irredundance of  $S$  also implies  $\langle [a, c] \rangle = \mathbb{Z}_2$ .

Furthermore, we may also assume that  $c$  either centralizes  $G'$  or inverts  $G'$ . (Otherwise, a preceding case applies after interchanging  $a$  with  $c$ .)

**Subsubcase 6.2.4.3.** Assume  $c$  inverts  $G'$ . Let

$$L = \begin{cases} (a, b)^{k\#} & \text{if } p \mid k \\ (b, a)^{k\#} & \text{if } p \nmid k \end{cases} \quad \text{and} \quad C = (L^{-1}, c^{-1}, L, c).$$

Then  $L$  is a hamiltonian path in  $\text{Cay}(\langle \bar{a} \rangle; a, b)$ , so  $C$  is a hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$ .

Since  $(ab)^k = \gamma^k$ , we have

$$\Pi L = \begin{cases} \gamma^k b^{-1} = \gamma^{k-1} a^{-k} & \text{if } p \mid k, \\ \gamma^{-k} a^{-1} & \text{if } p \nmid k. \end{cases}$$

Thus, in either case, we have  $\Pi L = \gamma^y a^z$ , where  $p \nmid y$  and  $z$  is odd, so

$$\begin{aligned} \Pi C &= (\Pi L)^{-1} c^{-1} (\Pi L) c = [\Pi L, c] = [\gamma^y a^z, c] \\ &= [\gamma^y, c]^{a^z} \cdot [a^z, c] = (\gamma^{-2y})^{a^z} \cdot [a, c]^z = \gamma^{2y} \cdot [a, c]. \end{aligned}$$

This generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subsubcase 6.2.4.4.** Assume  $c$  centralizes  $G'$  and  $k \geq 5$ . Let

$$C_0 = (L, c, L^{-1}, c^{-1}),$$

where  $L = (b, a)^{k\#}$ . Since  $C_0$  contains both  $[\bar{e}](b, a, b)$  and  $[\bar{a}\bar{b}\bar{c}](a^{-1})$ , and also contains both  $[\bar{a}^2](b, a, b)$  and  $[\bar{a}^3\bar{b}\bar{c}](a^{-1})$  we can apply Lemma 2.12 twice (first with  $s = b$ ,  $t = a$ ,  $u = c$ , and  $h = bc$ , and then with  $s = b$ ,  $t = a$ ,  $u = c$ , and  $h = a^2bc$ ), to obtain a hamiltonian cycle  $C_2$ , such that

$$(\Pi C_0)^{-1}(\Pi C_2) = [a^{-1}, b]^2,$$

which generates  $\mathbb{Z}_p$ . Then, since

$$\Pi C_0 = ((ba)^k a^{-1}) c ((ba)^k a^{-1})^{-1} c^{-1} = [a, c]$$

is a generator of  $\mathbb{Z}_2$ , we conclude that  $\Pi C_2$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subsubcase 6.2.4.5.** Assume  $c$  centralizes  $G'$  and  $k = 3$ . Assume, for the moment, that  $\gamma \notin \mathbb{Z}_p$ . Let

$$C = (c, b, c^{-1}, a, b^{-1}, c, b, a, b^{-1}, c^{-1}, b, a).$$



Then  $C$  is a hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$ , and a straightforward calculation shows that  $\Pi C = ba^3 = \gamma^{-1}$  generates  $G'$ , so the Factor Group Lemma (2.8) applies. [note A.45]

Now, suppose that  $p \geq 5$ , and, because of the preceding paragraph, that  $\gamma \in \mathbb{Z}_p$ . Let

$$C = (b, a, b^{-1}, a, b, c, a^{-5}, c^{-1}). \quad \text{[note A.46]}$$

Then  $C$  is a hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$  and

$$\Pi C = bab^{-1}abcac^{-1} = bab^{-1}aba[a, c^{-1}] = \gamma^{-3}[a, c]. \quad \text{[note A.47]}$$

Therefore  $\langle \Pi C \rangle = G'$  (since  $p \neq 3$  and  $\gamma$  projects trivially to  $\mathbb{Z}_2$ ), so the Factor Group Lemma (2.8) applies.

We may now assume  $p = 3$  (so  $|G| = 72$ ), and that  $\gamma \in \mathbb{Z}_p$ . Let  $\widehat{G} = G/\mathbb{Z}_p$ . We have the following hamiltonian cycle in  $\text{Cay}(\widehat{G}; S)$ :

$$C = (a^2, c, a^5, c^{-1}, a^{-2}, b, a^2, c, a^{-5}, c^{-1}, a^{-2}, b). \quad \text{[note A.48]}$$

Calculating modulo  $\mathbb{Z}_2$  (so  $c$  is in the center), we have

$$\Pi C = a^2ca^5c^{-1}a^{-2}ba^2ca^{-5}c^{-1}a^{-2}b \equiv a^2a^5a^{-2}ba^2a^{-5}a^{-2}b = a^{-1}bab = [a, b] = \gamma^2.$$

This is nontrivial (mod  $\mathbb{Z}_2$ ), so  $\Pi C$  must be nontrivial. Therefore  $\Pi C$  generates  $\mathbb{Z}_p$ , so the Factor Group Lemma (2.8) applies.

**Subcase 6.2.5.** Assume  $k = 2 < m/2$ .

**Subsubcase 6.2.5.1.** Assume  $|\overline{G} : \langle \overline{a} \rangle| > 2$ . Note that

$$C = (b, a, b^{-1}, c, b, a^{-1}, b, c^{-1}, (a, c, a, c^{-1})^{(m-4)/2}) \quad \text{[note A.49]}$$

is a cycle through the vertices of  $\text{Cay}(\overline{G}; \{a, b, c\})$  in  $\langle \overline{a} \rangle \cup c\langle \overline{a} \rangle$ . A connected sum of translates of  $C$  yields a hamiltonian cycle  $C_0$  in  $\text{Cay}(\overline{G}; S)$ . Since  $k$  is even, we know that  $\mathbb{Z}_2 \not\subseteq \langle [b, c] \rangle$ , so it is easy to choose the connected sum in such a way that  $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$  (see the proof of Lemma 3.15).

The cycle  $C$  contains the paths  $[\overline{e}](b, a, b^{-1})$  and  $[\overline{b}^2](a)$ . By taking just a bit of care in the creation of  $C_0$  (namely, not using any of these edges for the first connected sum), we may assume that  $C_0$  also contains these paths. Then Lemma 2.12 (with  $s = b, t = a, u = b$ , and  $h = b^2$ ) provides a hamiltonian cycle  $C_1$ , such that  $(\Pi C_0)^{-1}(\Pi C_1) = [a, b]^2$  (because  $b$  centralizes  $G'$ ). This is a generator of  $\mathbb{Z}_p$ , so either  $\Pi C_0$  or  $\Pi C_1$  generates  $G'$ . Therefore, the Factor Group Lemma (2.8) applies. [note A.50]

**Subsubcase 6.2.5.2.** Assume  $|\overline{G} : \langle \overline{a} \rangle| = 2$ . The irredundance of  $S$  implies that  $S = \{a, b, c\}$  (see Corollary 3.5). We have the following hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$ :

$$C = (b^2, a^{m-5}, c, a^{-(m-4)}, c^{-1}, b^{-1}, c, a, b^{-1}, c^{-1}). \quad \text{[note A.51]}$$

Since  $\overline{b} \in \langle \overline{a} \rangle$ , the irredundance of  $S$  implies  $\langle [a, c] \rangle = \mathbb{Z}_2$ . So  $m$  is even (see Corollary 3.8). [note A.37]  
However,  $\mathbb{Z}_2 \not\subseteq \langle [b, c] \rangle$ , because  $k = 2$  is even. So

$$\Pi C = b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \equiv b^2(a^{-1})(b^{-2}a[a, c]) \equiv [a, c] \pmod{\mathbb{Z}_p}, \quad \text{[note A.52]}$$

which generates  $\mathbb{Z}_2$ . We may also assume that  $c$  either centralizes  $G'$  or inverts  $G'$  (for otherwise a preceding case applies after interchanging  $a$  with  $c$ ). Therefore [note A.38]

$$\begin{aligned} \Pi C &= b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \equiv a^4\gamma^2(a^{-1})(\gamma^{-1}a^{-2}ca\gamma^{-1}a^{-2}c^{-1}) \\ &= \gamma^3 \cdot (\gamma^{-1})^c = \gamma^3 \cdot \gamma^{\pm 1} \in \{\gamma^2, \gamma^4\} \pmod{\mathbb{Z}_2}, \end{aligned} \quad \text{[note A.53]}$$

which generates  $\mathbb{Z}_p$ . We now know that  $\Pi C$  projects nontrivially to both  $\mathbb{Z}_2$  and  $\mathbb{Z}_p$ , so it generates  $G'$ . Therefore, the Factor Group Lemma (2.8) applies. □



## 7 Cases with $|\bar{a}| = 2$ and $\#S = 2$

**Assumption 7.1.** In this section, we assume

- $|\bar{a}| = 2$ , for all  $a \in S$ , such that  $a$  does not centralize  $G'$ , and
- $\#S = 2$ .

We may assume  $|a| = 2$ , for otherwise Case 4.1 applies with  $s = a$  and  $t = a^{-1}$ .

We may also assume that  $b$  centralizes  $G'$ , for otherwise we must have  $|\bar{b}| = 2$ , so  $|G| = 8p$ , so Theorem 2.3 applies. Since  $a$  does not centralize  $G'$ , this implies  $\bar{a} \notin \langle \bar{b} \rangle$ . Let

$$n = |\bar{G} : \langle \bar{a} \rangle| = |\bar{G}|/2 = |\bar{b}|.$$

**Case 7.2.** Assume  $n \not\equiv 1 \pmod{p}$ .

**Proof.** Let  $C = (a^{-1}, b^{-(n-1)}, a, b^{n-1})$ , so  $C$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$  with  $\Pi C = [a, b^{n-1}] = [a, b]^{n-1}$ , since  $b$  centralizes  $G'$ . Note that  $n$  is even (see Corollary 3.8), and, by assumption,  $n \not\equiv 1 \pmod{p}$ . Therefore,  $n-1$  is relatively prime to  $2p$ , so  $\Pi C$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.  $\square$

**Case 7.3.** Assume  $n \equiv 1 \pmod{p}$ .

**Proof.** We claim that  $\mathbb{Z}_p \subseteq \langle b \rangle$ . Suppose not. Then  $|\langle b, \mathbb{Z}_2 \rangle| = 2n$ . Since  $\gcd(2n, p) = 1$ , the abelian group  $\langle b, G' \rangle$  has a unique subgroup of order  $2n$ , so we conclude that  $\langle b, \mathbb{Z}_2 \rangle$  is normal in  $G$ . This implies that

$$\langle a \rangle \langle b, \mathbb{Z}_2 \rangle = \langle a, b, \mathbb{Z}_2 \rangle \supseteq \langle a, b \rangle = G,$$

so

$$|G| \leq |a| \cdot |\langle b, \mathbb{Z}_2 \rangle| = 2 \cdot 2n = 4n.$$

This contradicts the fact that  $|G| = 4np$ .

**Subcase 7.3.1.** Assume  $\mathbb{Z}_2 \subseteq \langle b \rangle$ . Combining this assumption with the above claim, we see that  $G' \subseteq \langle b \rangle$ . This implies  $\langle b \rangle \triangleleft G$ , so  $G = \langle a \rangle \rtimes \langle b \rangle$ . Since  $|a| = 2$ , this implies that  $\text{Cay}(G; a, b)$  is a generalized Petersen graph. Then the main result of [1] tells us that [note A.54]  $\text{Cay}(G; a, b)$  has a hamiltonian cycle.

**Subcase 7.3.2.** Assume  $\mathbb{Z}_2 \not\subseteq \langle b \rangle$ . Since  $\langle b, G' \rangle$  is abelian,  $\gcd(n, p) = 1$ , and  $\mathbb{Z}_2 \not\subseteq \langle b \rangle$ , we may write

$$\langle b, G' \rangle = \mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_n.$$

Then  $G = \langle a \rangle \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_n)$ , and we may assume  $b = (0, 1, 1)$  and  $[a, b] = (1, 2, 0)$ . For  $\underline{G} = G/\langle b^2 \rangle = G/(\mathbb{Z}_p \times 2\mathbb{Z}_n)$ , it is straightforward to check that  $((a, b)^{4\#}, b^{-1})$  is a [note A.55] hamiltonian cycle in  $\text{Cay}(\underline{G}; a, b)$  whose voltage is  $(0, -2, 2)$ . (This hamiltonian cycle is taken from the final paragraph of Case 1 of the proof of [3, Prop. 6.1].) This voltage generates  $\mathbb{Z}_p \times 2\mathbb{Z}_n$  (since  $\gcd(p, n) = 1$ ), so the Factor Group Lemma (2.8) applies.  $\square$

## 8 Cases with $|\bar{a}| = 2$ and $\#S = 3$

**Assumption 8.1.** In this section, we assume

$$S = \{a, b, c\},$$

and

$$|\bar{s}| = 2, \text{ for all } s \in S, \text{ such that } s \text{ does not centralize } G'.$$

We also assume Case 4.1 does not apply. (So  $|s| = 2$ .) In particular, we have  $|a| = 2$ .

Note that  $\bar{a} \notin \langle \bar{b} \rangle$ . (If  $\bar{a} \in \langle \bar{b} \rangle$ , then  $b$ , like  $a$ , does not centralize  $G'$ , so our assumption implies  $|\bar{b}| = 2$ . Then  $\bar{a} = \bar{b}$ , contradicting the fact that Case 4.1 does not apply.)

**Notation 8.2.** Let

$$n = |\bar{b}| = |\langle \bar{a}, \bar{b} \rangle : \langle \bar{a} \rangle| \geq 2 \quad \text{and} \quad \ell = |\bar{G} : \langle \bar{a}, \bar{b} \rangle| = |\bar{G}|/(2n) \geq 2.$$

The last inequality is because the irredundance of  $S$  implies  $\bar{c} \notin \langle \bar{a}, \bar{b} \rangle$  (see Corollary 3.5).

**Case 8.3.** Assume  $|\bar{b}| = 3$ .

**Proof.** Since  $|\bar{b}| \neq 2$ , Assumption 8.1 implies that  $b$  centralizes  $G'$ . Also, since  $|\bar{b}|$  is odd, Corollary 3.8 implies that  $[a, b]$  and  $[b, c]$  project trivially to  $\mathbb{Z}_2$ , so  $[a, c]$  must project nontrivially (and  $\ell$  must be even). We have the following hamiltonian path in  $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; S)$ :

$$L = (c^{\ell-1}, b, c^{-(\ell-1)}, b, c^{\ell-1}). \quad [\text{note A.56}]$$

Then  $C = (L, a, L^{-1}, a)$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ . Since  $\ell - 1$  is odd, it is easy to see that  $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$ . [note A.57]

Since  $C$  contains both  $[\bar{c}^{\ell-2}](c, b, c^{-1})$  and  $[\bar{c}^{\ell-1}\bar{a}\bar{b}](b^{-1})$ , Lemma 2.12 (with  $s = c$ ,  $t = b$ ,  $u = a$ , and  $h = c^{\ell-1}a$ ) provides a hamiltonian cycle  $C'$ , such that  $(\Pi C)^{-1}(\Pi C')$  is conjugate to  $[t^{-1}, u][s, t^{-1}]^u = [b^{-1}, a][c, b^{-1}]^a = [a, b][c, b]$ . This is an element of  $\mathbb{Z}_p$ . If it generates  $\mathbb{Z}_p$ , then either  $\Pi C$  or  $\Pi C'$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

Thus, we may assume  $[a, b][c, b]$  is trivial. Since  $\mathbb{Z}_p \subseteq \langle [a, b] \rangle$  (see (3.3B)), this implies that  $[c, b]$  is nontrivial. So we may assume that  $c$  does not centralize  $\mathbb{Z}_p$  (for otherwise replacing  $c$  with  $c^{-1}$  would replace  $[c, b]$  with  $[c, b]^{-1}$ , which would not cancel  $[a, b]$ ).

Now, Assumption 8.1 implies  $|\bar{c}| = 2$ , so we have the hamiltonian cycle

$$C_0 = (b^2, a, b^2, c, a, b, a, b, a, c), \quad [\text{note A.58}]$$

in  $\text{Cay}(\bar{G}; S)$ . This contains both the path  $[bac](a, b, a)$  and the edge  $[b](b)$ , so applying Lemma 2.12 (with  $s = a$ ,  $t = b$ ,  $u = c$ , and  $h = b$ ) provides a hamiltonian cycle  $C_1$ , such that  $(\Pi C_0)^{-1}(\Pi C_1)$  is conjugate to  $[u, t^{-1}][s, t^{-1}]^u = [c, b^{-1}][a, b^{-1}]^c$ . This is not equal to  $[a, b][c, b]$  (which is trivial), because  $[a, b^{-1}]^c = [a, b]$ , but  $[c, b^{-1}] = [c, b]^{-1} \neq [c, b]$ . So  $(\Pi C_0)^{-1}(\Pi C_1)$  is nontrivial, and therefore generates  $\mathbb{Z}_p$ . Since a straightforward calculation shows that  $\mathbb{Z}_2$  is contained in  $\langle \Pi C_0 \rangle$ , this implies that either  $\Pi C_0$  or  $\Pi C_1$  generates  $G'$ , so the Factor Group Lemma (2.8) applies. [note A.59]  $\square$

**Case 8.4.** Assume  $\ell = 2$ .

**Proof.** We may assume  $|\bar{b}| \geq 4$ , for otherwise either  $|\bar{b}| = 2$ , so Theorem 2.3 applies (because  $|G| = 16p$ ), or  $|\bar{b}| = 3$ , so Case 8.3 applies. Let

$$L = (a, b, a, b^{n-2}, a, b^{-(n-3)}) \quad \text{and} \quad C = (L, c, L^{-1}, c^{-1}), \quad [\text{note A.60}]$$

so  $L$  is a hamiltonian path in  $\text{Cay}(\langle \bar{a}, \bar{b} \rangle; a, b)$  and  $C$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ .

**Subcase 8.4.1.** Assume  $[a, c]$  and  $[a, b][b, c]$  are not both in  $\mathbb{Z}_p$ . A straightforward calculation (using Lemma 3.6) shows that  $\Pi C \equiv [a, c] \pmod{\mathbb{Z}_p}$ . If this is in  $\mathbb{Z}_p$ , then, [note A.61] by assumption,  $[a, b][b, c] \notin \mathbb{Z}_p$ , so applying Lemma 2.12 to the paths  $[\bar{c}](a, b, a)$  and  $[\overline{abc}](b^{-1})$  in  $C$  (so  $s = a$ ,  $t = b$ ,  $u = c$ , and  $h = ac$ ) yields a hamiltonian cycle  $C'$ , such that  $\Pi C'$  projects nontrivially to  $\mathbb{Z}_2$ . Therefore, we have a hamiltonian cycle (either  $C$  [note A.62] or  $C'$ ) whose voltage is not in  $\mathbb{Z}_p$ .

Now, since  $|\bar{b}| \geq 4$ , we know that  $C$  (and also  $C'$ ) contains the paths  $[\overline{b^{-2}ac}](b, a, b^{-1})$  and  $[\overline{ac}](a)$ . Furthermore, we know that  $[b, a][b, a]^b$  is a nontrivial element of  $\mathbb{Z}_p$  (because  $b$  does not invert  $[a, b]$ ). Therefore, Lemma 2.12 (with  $s = b, t = a, u = b$ , and  $h = ac$ ) yields a hamiltonian cycle  $C_1$  (or  $C'_1$ ) whose voltage generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 8.4.2.** Assume  $[a, c]$  and  $[a, b][b, c]$  are both in  $\mathbb{Z}_p$ . Since  $[a, c]$ ,  $[a, b]$ , and  $[b, c]$  generate  $G'$ , they cannot all be in  $\mathbb{Z}_p$ , so this assumption implies that neither  $[a, b]$  nor  $[b, c]$  is in  $\mathbb{Z}_p$ . Also, we may assume  $\langle [a, c] \rangle = \mathbb{Z}_p$ , for otherwise  $[a, c] = e$ , so we could apply Lemma 2.13 with  $s = c$ .

We have the following hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ :

[note A.63]

$$C_0 = (b^{n-1}, c, b^{-(n-2)}, a, b^{n-2}, c^{-1}, b^{-(n-1)}, c, a, c^{-1}).$$

Then

$$\begin{aligned} \Pi C_0 &= b^{n-1} c (b^{-(n-2)} a b^{n-2}) c^{-1} b^{-(n-1)} c a c^{-1} \\ &= b^{n-1} c (a [a, b]^{n-2}) c^{-1} b^{-(n-1)} c a c^{-1} \\ &= ([a, b]^{-(n-2)})^c \cdot b^{n-1} (c a c^{-1}) b^{-(n-1)} (c a c^{-1}) \\ &= ([a, b]^{-(n-2)})^c \cdot [b, c a c^{-1}]^{-(n-1)} \\ &= ([a, b]^{-(n-2)})^c \cdot [b, a]^{-(n-1)} \quad (c a c^{-1} \in a G' \text{ and } G' \subseteq C_G(b)) \\ &= ([a, b]^{-(n-2)})^c \cdot [a, b]^{n-1}. \end{aligned}$$

If  $c$  centralizes  $\mathbb{Z}_p$ , then  $\Pi C_0 = [a, b]$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

We may now assume  $c$  does not centralize  $\mathbb{Z}_p$ . Then Assumption 8.1 tells us that  $c$  inverts  $\mathbb{Z}_p$ , so  $\Pi C_0 = [a, b]^{2n-3}$  (and  $|c| = 2$ ). Hence, we may assume  $2n \equiv 3 \pmod{p}$ , for otherwise  $\Pi C_0$  generates  $G'$ , so the Factor Group Lemma (2.8) applies. We now consider the following hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$ :

[note A.64]

$$C_* = (b^{n-3}, c, b^{-(n-4)}, a, b^{n-4}, c^{-1}, b^{-(n-3)}, c, (b^{-1}, c)^2, a, (c, b)^2, c^{-1}).$$

We have

$$\Pi C_* = b^{n-3} c (b^{-(n-4)} a b^{n-4}) c^{-1} b^{-(n-3)} c ((b^{-1} c)^2 a (c b)^2) c^{-1}.$$

[note A.65]

Since  $cb$  inverts  $G'$ , we know that  $(b^{-1} c)^2 a (c b)^2 = a$ , so  $\Pi C_*$  is exactly the same as the voltage of  $C_0$ , but with  $n$  replaced by  $n - 2$ ; that is,

$$\Pi C_* = [a, b]^{2(n-2)-3} = [a, b]^{2n-7}.$$

Since  $2n \equiv 3 \pmod{p}$ , we have

$$2n - 7 \equiv 3 - 7 = -4 \not\equiv 0 \pmod{p},$$

so  $\Pi C_*$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.  $\square$

**Case 8.5.** Assume  $|\bar{b}| \neq 3$  and  $\ell \neq 2$ .

**Proof.** Since  $\ell \neq 2$ , we know  $|\bar{c}| > 2$ , so  $c$  must centralize  $G'$  (by Assumption 8.1). Also, Corollary 3.8 implies that  $|\bar{b}|$  and  $\ell$  cannot both be odd.

- If  $|\bar{b}|$  is odd (so  $\ell$  is even), let

[note A.66]

$$L = (c^{\ell-1}, b, c^{-1}, b, c, b, (b^{n-4}, c^{-1}, b^{-(n-4)}, c^{-1})^{\ell/2}, b^{-1}, c^{\ell-3}, b^{-1}, c^{-(\ell-3)}).$$

- If  $|\bar{b}|$  is even, let

$$L = (c^{\ell-1}, b^{n-1}, c^{-1}, (c^{-(\ell-2)}, b^{-1}, c^{\ell-2}, b^{-1})^{(n-2)/2}, c^{-(\ell-2)}). \quad [\text{note A.67}]$$

In either case,  $L$  is a hamiltonian path in  $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; \{b, c\})$  from  $\bar{e}$  to  $\bar{b}$ . Now, let

$$C = (L, a, L^{-1}, a) \quad \text{and} \quad (g, \varepsilon) = \begin{cases} (c^{\ell-1}, -1) & \text{if } |\bar{b}| = 2 \text{ or } |\bar{b}| \text{ is odd,} \\ (ab^2, 1) & \text{if } |\bar{b}| > 2 \text{ and } |\bar{b}| \text{ is even,} \end{cases}$$

so  $C$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S)$  that contains the paths

$$[\bar{bc}](c^{-1}, a, c), \quad [\bar{ca}](c^{-1}, a, c), \quad [\bar{g}](b), \quad \text{and} \quad [\bar{gbac}^\varepsilon](c^{-\varepsilon}, b^{-1}, c^\varepsilon).$$

Note that  $[\bar{bc}](c^{-1}, a, c)$  contains  $[\bar{b}](a)$  and that  $[\bar{ca}](c^{-1}, a, c)$  contains  $[\bar{a}](a)$ . Also note that all of these paths are vertex-disjoint (except for the vertices  $\bar{ac}$  and  $\{abc\}$  when  $|\bar{b}| = 2$  and  $\ell = 3$ ). We introduce some terminology:

- Applying Lemma 2.12 to the oriented paths  $[\bar{ca}](c^{-1}, a, c)$  and  $[\bar{b}](a)$  (so  $s = c^{-1}$ ,  $t = a$ ,  $u = b$ , and  $h = ab$ ) will be called the “ $a$ -transform.” This multiplies the voltage by  $\gamma_a$ , where  $\gamma_a = [a, b^{-1}][c, a]$ . [note A.68]
- Applying Lemma 2.12 to the oriented paths  $[\bar{g}](b)$  and  $[\bar{gbac}^\varepsilon](c^{-\varepsilon}, b^{-1}, c^\varepsilon)$  (so  $s = c^{-\varepsilon}$ ,  $t = b^{-1}$ ,  $u = a$ , and  $h = gb$ ) will be called the “ $b$ -transform.” This multiplies the voltage by a conjugate of  $\gamma_b$ , where  $\gamma_b = [b, a][b, c^{-\varepsilon}]$ . [note A.69]

**Subcase 8.5.1.** Assume precisely one of  $\gamma_a$  and  $\gamma_b$  is in  $\mathbb{Z}_p$ . Write  $\{a, b\} = \{x, y\}$ , such that  $\gamma_x \in \mathbb{Z}_p$  and  $\gamma_y \notin \mathbb{Z}_p$ . We may assume  $\langle \gamma_x \rangle = \mathbb{Z}_p$  (by replacing  $c$  with its inverse, if necessary). Choose  $C'$  to be either  $C$  or the  $y$ -transform of  $C$ , such that  $\Pi C'$  projects nontrivially to  $\mathbb{Z}_2$ . Then choose  $C''$  to be either  $C'$  or the  $x$ -transform of  $C'$ , such that  $\Pi C''$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 8.5.2.** Assume  $\gamma_a$  and  $\gamma_b$  are both in  $\mathbb{Z}_p$ . Since  $[a, b]$ ,  $[a, c]$ , and  $[b, c]$  cannot all be in  $\mathbb{Z}_p$ , this assumption implies that none of them are in  $\mathbb{Z}_p$ . Therefore, since the path  $L$  has odd length, we see that  $\Pi C$  has nontrivial projection to  $\mathbb{Z}_2$ . [note A.70]

We may assume (by replacing  $c$  with its inverse, if necessary), that  $\gamma_a$  has nontrivial projection to  $\mathbb{Z}_p$ , so  $\langle \gamma_a \rangle = \mathbb{Z}_p$ . Therefore, by choosing  $C'$  to be either  $C$  or the  $a$ -transform of  $C$ , such that  $\Pi C'$  generates  $G'$ , we may apply the Factor Group Lemma (2.8).

**Subcase 8.5.3.** Assume neither  $\gamma_a$  nor  $\gamma_b$  is in  $\mathbb{Z}_p$ , and  $b$  centralizes  $G'$ . Note that the sum of the exponents of the occurrences of  $b$  in  $L$  is 1, and the sum of the exponents of the occurrences of  $c$  is 0. Therefore, since  $b$  and  $c$  centralize  $G'$ , Lemma 3.6 implies that  $\Pi C = [a, b]$ . Hence, we may assume  $[a, b] \in \mathbb{Z}_p$  (for otherwise  $\langle \Pi C \rangle = G'$ , so the Factor Group Lemma (2.8) applies). Then, by the assumption of this subcase, we conclude that  $[a, c] \notin \mathbb{Z}_p$ . So we may assume  $\langle [a, c] \rangle = \mathbb{Z}_2$ , for otherwise  $b$  and  $c$  could be interchanged, resulting in a situation in which  $[a, b] \notin \mathbb{Z}_p$ , and which has therefore already been covered. Also, since  $[a, b] \in \mathbb{Z}_p$  and  $[a, c] \notin \mathbb{Z}_p$ , Corollary 3.8 tells us that  $\ell$  is even (and recall that  $\ell \neq 2$ ). [note A.71]

Since  $[a, b]$  is a nontrivial element of  $\mathbb{Z}_p$ , and  $b$  centralizes  $G'$ , we see from Corollary 3.7 that  $|b|$  is divisible by  $p$ . Therefore,  $|b| \neq 2$ , so we may assume  $|\bar{b}| > 2$  (for otherwise Case 4.1 applies with  $s = b$  and  $t = b^{-1}$ ). Since  $|\bar{b}| \neq 3$  (by the assumption of this case), this implies  $n = |\bar{b}| \geq 4$ , so we may let [note A.72]

$$L_0 = (c^{\ell-1}, b, c^{-(\ell-1)}, b^2, (b^{n-4}, c, b^{-(n-4)}, c)^{\ell/2}, b^{-1}, c^{-(\ell-2)}),$$

so  $L_0$  is a hamiltonian path from  $\bar{e}$  to  $\bar{b}^2c$  in  $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; \{b, c\})$ . Note that the sum of the exponents of the occurrences of  $b$  in  $L$  is 2, and the sum of the exponents of

the occurrences of  $c$  is 1. Therefore, since  $b$  and  $c$  centralize  $G'$ , Lemma 3.6 implies  $\Pi(L_0, a, L_0^{-1}, a) = [a, b]^2[a, c]$ . This generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 8.5.4.** Assume neither  $\gamma_a$  nor  $\gamma_b$  is in  $\mathbb{Z}_p$ , and  $b$  does not centralize  $\mathbb{Z}_p$ . From Assumption 8.1, we know  $\bar{b} = 2$  (so  $b$  must invert  $G'$ ).

We may assume  $[a, c] \in \mathbb{Z}_2$ , for otherwise Case 8.4 could be applied by interchanging  $b$  and  $c$ . Then we may assume  $[a, c]$  is the nontrivial element of  $\mathbb{Z}_2$ , for otherwise the assumption that  $\gamma_a \notin \mathbb{Z}_p$  implies  $\langle [a, b] \rangle = G'$ , so  $\langle a, b \rangle \triangleleft G$ , and then Lemma 2.13 applies with  $s = c$ .

By applying the same argument, with  $a$  and  $b$  interchanged, we may assume  $[b, c]$  is also the nontrivial element of  $\mathbb{Z}_2$ . This implies  $[a, b] \in \mathbb{Z}_p$ , since  $\gamma_b \notin \mathbb{Z}_p$ .

Note that, since  $a$  and  $b$  both have order 2 (and invert  $G'$ ), the image of  $\langle a, b \rangle$  in  $G/\mathbb{Z}_2$  is the dihedral group of order  $2p$ . Also, the preceding two paragraphs imply that  $c$  is in the center of  $G/\mathbb{Z}_2$ . Therefore, we have the following hamiltonian cycle in  $\text{Cay}(G/\mathbb{Z}_2; S)$ :

[note A.73]

$$C = (c, (c^{\ell-2}, a, c^{-(\ell-2)}, b)^{p\#}, c^{-1}, (a^{-1}, b^{-1})^{p\#}).$$

Since  $[a, b]$  projects trivially to  $\mathbb{Z}_2$ , Corollary 3.8 implies that  $\ell$  is even, so, calculating modulo  $\mathbb{Z}_p$ , we have

$$\begin{aligned} \Pi C &= c(c^{\ell-2}ac^{-(\ell-2)}b)^pb^{-1}c^{-1}(a^{-1}b^{-1})^pb \\ &\equiv c(ab)^pb^{-1}c^{-1}(a^{-1}b^{-1})^pb && \left( \begin{array}{l} \ell-2 \text{ is even, so } c^{\ell-2} \\ \text{is central modulo } \mathbb{Z}_p \end{array} \right) \\ &\equiv z^{2p-1}(ab)^pb^{-1}(a^{-1}b^{-1})^pb && \left( \begin{array}{l} \text{letting } z = [a, c] = [b, c] \text{ be} \\ \text{the nontrivial element of } \mathbb{Z}_2 \end{array} \right) \\ &\equiv z && (z^2 = e \text{ and } [a, b] \in \mathbb{Z}_p). \end{aligned}$$

Since this generates  $\mathbb{Z}_2$ , the Factor Group Lemma (2.8) applies.  $\square$

## 9 Cases with $|\bar{a}| = 2$ and $\#S \geq 4$

**Assumption 9.1.** In this section, we assume

- $\#S \geq 4$ , and
- $|\bar{s}| = 2$ , for all  $s \in S$ , such that  $s$  does not centralize  $G'$ .

We also assume Case 4.1 does not apply. (So  $|s| = 2$ .)

[note A.74] Furthermore, we assume  $\bar{b} \notin \langle \bar{a} \rangle$  (otherwise, Case 4.1 applies). Then it is easy that we also have  $\bar{a} \notin \langle \bar{b} \rangle$ .

**Outline.** This final section of the proof is longer than the others, so here is an outline of the cases and subcases that it considers.

9.4: Assume no element of  $S$  centralizes  $G'$ .

9.4.1: Assume  $\#S \geq 5$ .

9.4.2: Assume  $\#S = 4$ .

9.5: Assume there exists  $s \in S$ , such that  $[a, s] \notin \mathbb{Z}_p$ , and, in addition, either  $s = b$ , or  $b$  centralizes  $G'$ , or  $\mathbb{Z}_p \subseteq \langle S \setminus \{a\} \rangle'$ .

9.5.1: Assume  $\mathbb{Z}_p \not\subseteq \langle S \setminus \{a\} \rangle'$ .

9.5.2: Assume  $\mathbb{Z}_p \subseteq \langle S \setminus \{a\} \rangle'$ .

9.6: Assume  $b$  centralizes  $G'$ .

9.6.1: Assume there exists  $c \in S$ , such that  $[c, b] \notin \mathbb{Z}_p$ .

**9.6.2:** Assume  $[c, b] \in \mathbb{Z}_p$  for all  $c \in S$ .

**9.7:** Assume that none of the preceding cases apply.

Since Case 9.4 does not apply, some element  $c$  of  $S$  centralizes  $G'$ .

**9.7.1:** Assume  $\langle [s, c] \rangle \neq \mathbb{Z}_2$ , for some  $s \in S \setminus \{c\}$ .

**9.7.2:** Assume  $\langle [s, c] \rangle = \mathbb{Z}_2$ , for all  $s \in S \setminus \{c\}$ .

**Notation 9.2.** Let  $n = |\bar{b}|$  and  $\ell = |\bar{G} : \langle \bar{a}, \bar{b} \rangle| = |\bar{G}|/(2n)$ .

**Note 9.3.** The irredundance of  $S$  implies  $S \setminus \{a, b\}$  is an irredundant generating set for  $\bar{G}/\langle \bar{a}, \bar{b} \rangle$  (see Corollary 3.5), so  $\ell \geq 4$ .

**Case 9.4.** Assume no element of  $S$  centralizes  $G'$ .

**Proof.** From Assumption 9.1, we see that every element of  $S$  inverts  $G'$  (and has order 2). We may assume no two elements of  $S$  commute, for otherwise it is not difficult to see that Lemma 2.13 applies. [note A.75]

Let  $c, d \in S \setminus \{a, b\}$ , and let  $\gamma = [a, b][a, c]$ . We claim that we may assume  $\gamma \notin \mathbb{Z}_2$ , by permuting  $b, c, d$ . To this end, first note that if  $\gamma \in \mathbb{Z}_2$ , then  $\mathbb{Z}_p \subseteq \langle [a, c] \rangle$ , so there is no harm in putting  $c$  into the role of  $b$ . Now, let us suppose  $[a, b][a, c]$ ,  $[a, c][a, d]$ , and  $[a, d][a, b]$  are all in  $\mathbb{Z}_2$ . Then

$$[a, b] \equiv [a, c]^{-1} \equiv [a, d] \equiv [a, b]^{-1} \pmod{\mathbb{Z}_2},$$

which contradicts the fact that  $[a, b] \notin \mathbb{Z}_2$  (and  $p$  is odd).

Let

$$C = ((c, a, c, b)^2 \#, d)^2,$$

[note A.76]

so  $C$  is a hamiltonian cycle in  $\text{Cay}(\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle; \{a, b, c, d\})$  that contains the vertex-disjoint paths  $[\bar{e}](c, a, c)$ ,  $[\overline{abc}](a)$ ,  $[\overline{bd}](c, a, c)$ , and  $[\overline{acd}](a)$ . Applying Lemma 2.12 to the paths  $[\bar{e}](c, a, c)$  and  $[\overline{abc}](a)$  (so  $s = c$ ,  $t = a$ ,  $u = b$ , and  $h = bc$ ) will multiply the voltage by  $\gamma$ . Applying Lemma 2.12 to the other two paths  $[\overline{bd}](c, a, c)$  and  $[\overline{acd}](a)$  (so  $s = c$ , [note A.77]  
 $t = a$ ,  $u = b$ , and  $h = cd$ ) will also multiply the voltage by  $\gamma$  (because  $bc$  and  $cd$  both [note A.78]  
centralize  $G'$ ). Therefore, applying Lemma 2.12 twice yields a hamiltonian cycle  $C''$ , such that  $(\Pi C)^{-1}(\Pi C'') = \gamma^2$ , which is a generator of  $\mathbb{Z}_p$ .

**Subcase 9.4.1.** Assume  $\#S \geq 5$ . If there exist  $s, t \in S$ , such that  $s \notin \{a, b, c\}$ , and  $[s, t] \notin \mathbb{Z}_p$ , then the preceding paragraph implies that Lemma 3.15(2) applies.

Thus, we may assume that the preceding condition does not apply (for any legitimate choice of  $a, b$ , and  $c$ ). Fix two elements  $x, y \in S \setminus \{a, b, c\}$ . The failure of the condition implies  $[x, S] \subseteq \mathbb{Z}_p$ . In particular,  $[x, y]$  must be a generator of  $\mathbb{Z}_p$  (because no two elements of  $S$  commute), so we may let  $\{x, y\}$  play the role of  $\{a, b\}$ . So we may let  $\{x, y, b, c\}$  play the role of  $\{a, b, c, d\}$ . Then, since  $a \notin \{x, y, b, c\}$ , the failure of the condition implies  $[a, S] \subseteq \mathbb{Z}_p$ . Similarly,  $[b, S]$  and  $[c, S]$  are also in  $\mathbb{Z}_p$ . So  $[s, t] \subseteq \mathbb{Z}_p$  for all  $s, t \in S$ . This contradicts the fact that  $\langle [S, S] \rangle = G' \not\subseteq \mathbb{Z}_p$ .

**Subcase 9.4.2.** Assume  $\#S = 4$ . For convenience, in this subcase (and only in this subcase), we drop our standing assumption that  $\langle [a, b] \rangle$  contains  $\mathbb{Z}_p$ . Instead, choose  $b, d \in S$ , such that  $[b, d]$  projects nontrivially to  $\mathbb{Z}_2$ . A straightforward calculation (using the fact that  $a, b, c$ , and  $d$  all invert  $G'$ ) shows that

$$\Pi C = [c, d]^4 [d, a]^2 [d, b].$$

[note A.79]

Since  $[d, b]$  projects nontrivially to  $\mathbb{Z}_2$ , but  $[c, d]^4$  and  $[d, a]^2$  have even exponents, so they obviously do not, we see that  $\mathbb{Z}_2 \subseteq \langle \Pi C \rangle$ . Therefore, we may assume  $\Pi C \in \mathbb{Z}_2$ , for otherwise the Factor Group Lemma (2.8) applies.

We may assume  $\gamma \in \mathbb{Z}_2$ , for otherwise applying Lemma 2.12 twice (as in the paragraph immediately before Subcase 9.4.1) yields a hamiltonian cycle whose voltage generates  $G'$ , so the Factor Group Lemma (2.8) applies. By the definition of  $\gamma$ , this means  $[a, b][a, c] \in \mathbb{Z}_2$ . And we may assume the same is true when  $b$  and  $d$  are interchanged, which means  $[a, d][a, c] \in \mathbb{Z}_2$ . So

$$[a, b] \equiv [a, c]^{-1} \equiv [a, d] \pmod{\mathbb{Z}_2}.$$

By interchanging  $a$  and  $c$ , we conclude that we may also assume

$$[c, b] \equiv [c, a]^{-1} \equiv [c, d] \pmod{\mathbb{Z}_2}.$$

So

$$[c, d] \equiv [c, a]^{-1} = [a, c] \equiv [a, d]^{-1} = [d, a] \pmod{\mathbb{Z}_2}.$$

Therefore

$$[d, a]^6 [d, b] = [d, a]^4 [d, a]^2 [d, b] \equiv [c, d]^4 [d, a]^2 [d, b] = \Pi C \equiv 0 \pmod{\mathbb{Z}_2}.$$

If  $p \neq 3$ , then, since we may assume the same is true when we interchange  $a$  and  $c$ , [note A.80] we conclude that  $[d, c] \equiv [d, a] \pmod{\mathbb{Z}_2}$ . Since we also have  $[c, d] \equiv [d, a] \pmod{\mathbb{Z}_2}$ , we conclude that  $[c, d]$  and  $[a, d]$  are in  $\mathbb{Z}_2$ . This implies  $[b, d] \notin \mathbb{Z}_2$  (since  $d$  does not centralize  $\mathbb{Z}_p$ , and is therefore not in the center of  $G/\mathbb{Z}_2$ ), so

$$\Pi C = [c, d]^4 [a, d]^2 [d, b] \equiv e^4 e^2 [d, b] = [d, b] \not\equiv 0 \pmod{\mathbb{Z}_2}.$$

This contradicts the fact that  $\Pi C \in \mathbb{Z}_2$ .

We now assume  $p = 3$ . Then the equation  $[d, a]^6 [d, b] \equiv 0 \pmod{\mathbb{Z}_2}$  implies  $[d, b] \in \mathbb{Z}_2$ . This conclusion came from assuming only that  $[d, b] \notin \mathbb{Z}_p$ . Therefore, for all  $s, t \in S$ , the commutator  $[s, t]$  must be in either  $\mathbb{Z}_2$  or  $\mathbb{Z}_p$ . However,

$$[a, b] \equiv [c, a] \equiv [a, d] \equiv [b, c] \equiv [d, c] \pmod{\mathbb{Z}_2},$$

and  $[a, b] \notin \mathbb{Z}_2$ . Therefore, we conclude all five of these other commutators are in  $\mathbb{Z}_p$ . (Therefore, the stated congruences between these commutators are actually equalities.)

Now, interchanging  $a \leftrightarrow b$  and  $c \leftrightarrow d$  in  $C$  yields a hamiltonian cycle  $C^*$ , such that

$$\Pi C^* = [d, c]^4 [c, b]^2 [c, a] = [d, c][b, c][c, a] = [c, a]^3 = e$$

(because  $p = 3$ ). Let  $\gamma^* = [b, a][b, d]$ , so  $\gamma^*$  is obtained from  $\gamma = [a, b][a, c]$  by interchanging  $a \leftrightarrow b$  and  $c \leftrightarrow d$ . Then, since applying Lemma 2.12 to  $C$  can multiply the voltage by  $\gamma = [a, b][a, c]$ , we know that applying Lemma 2.12 to  $C^*$  can multiply the voltage by  $\gamma^*$ , which generates  $G'$ . So the Factor Group Lemma (2.8) applies.  $\square$

**Case 9.5.** Assume there exists  $s \in S$ , such that  $[a, s] \notin \mathbb{Z}_p$ , and:

$$\text{either } s = b, \text{ or } b \text{ centralizes } G', \text{ or } \mathbb{Z}_p \subseteq \langle S \setminus \{a\} \rangle'.$$

**Proof.** Let  $S_0 = S \setminus \{a\}$ . Note that the irredundance of  $S$  implies  $a \notin \langle S_0 \rangle \mathbb{Z}_2$  (see Lemma 3.4).

**Subcase 9.5.1.** Assume  $\mathbb{Z}_p \not\subseteq \langle S_0 \rangle'$ . If  $[a, b] \notin \mathbb{Z}_p$ , we assume that  $s = b$ . Let

$$g = \begin{cases} s & \text{if } [s, a] \notin \mathbb{Z}_2, \\ sb^2 & \text{if } [s, a] \in \mathbb{Z}_2. \end{cases}$$

[note A.81] Note that  $\langle [g, a] \rangle = G'$ .

Let  $H^* = \langle S_0 \rangle \mathbb{Z}_2 / \mathbb{Z}_2$ . From the assumption of this subcase, we know that  $H^*$  is abelian. Therefore, Corollary 2.11 provides a hamiltonian path  $L = (s_i)_{i=1}^r$  in  $\text{Cay}(\overline{H^*}; S_0)$ , such that  $s_1 s_2 \cdots s_r \in g \mathbb{Z}_2$ . Then  $(L^{-1}, a, L, a)$  is a hamiltonian cycle in  $\text{Cay}(\overline{G}; S)$ , and

$$\Pi C = [s_1 s_2 \cdots s_r, a] \in [g \mathbb{Z}_2, a] = \{[g, a]\}$$

(since  $\mathbb{Z}_2$  is in the center of  $G$ ). This voltage generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 9.5.2.** Assume  $\mathbb{Z}_p \subseteq \langle S_0 \rangle'$ . Suppose  $w, x, y \in S^{\pm 1} \setminus \{a\}$ , such that

$$\langle \bar{w} \rangle \subsetneq \langle \bar{w}, \bar{x} \rangle \subsetneq \langle \bar{w}, \bar{x}, \bar{y} \rangle. \quad (9.5A)$$

It is easy to construct a hamiltonian cycle  $C_0$  in  $\text{Cay}(\langle \overline{S_0} \rangle; S_0)$ , such that  $C_0$  contains the oriented paths  $[\overline{hw^{-1}y^{-1}}](w, x, w^{-1})$  and  $[\overline{hx}](x^{-1})$ , for some  $h \in G$ . Furthermore, if [note A.82]

$$\text{either } x \notin \{s^{\pm 1}\} \text{ or } |\overline{G}| > 16, \quad (9.5B)$$

then, for some  $\varepsilon \in \{\pm 1\}$ , it is not difficult to arrange that the hamiltonian cycle  $C_0$  contains the oriented edge  $[s^\varepsilon](s^{-\varepsilon})$ , and that this edge is not in either of the above-mentioned paths. [note A.83]

Applying Lemma 2.12 to the first two paths (so  $s = w$ ,  $t = x$ , and  $u = y$ ) yields a hamiltonian cycle  $C_1$ , such that  $(\Pi C_0)^{-1}(\Pi C_1)$  is conjugate to  $[x^{-1}, y][w, x^{-1}]^y$ . Removing the edge  $[s^\varepsilon](s^{-\varepsilon})$  yields hamiltonian paths  $C_0\#$  and  $C_1\#$  from  $\bar{e}$  to  $\bar{s}^\varepsilon$ .

From Lemma 3.4 and the assumption of this subcase, we see that  $\langle \overline{S_0} \rangle \neq \overline{G}$ . So [note A.84]

$$C_0^+ = (C_0\#, a, (C_0\#)^{-1}, a) \text{ and } C_1^+ = (C_1\#, a, (C_1\#)^{-1}, a)$$

are hamiltonian cycles in  $\text{Cay}(\overline{G}; S)$ . For  $k = 0, 1$ , we have

$$\Pi C_k^+ = [((\Pi C_k)s^\varepsilon)^{-1}, a].$$

Since  $\Pi C_k \in G'$ , and  $G'$  is central modulo  $\mathbb{Z}_p$  (and from the choice of  $s$ ), we have

$$\Pi C_k^+ \equiv [s^\varepsilon, a] \not\equiv e \pmod{\mathbb{Z}_p}.$$

Furthermore, if  $[x^{-1}, y][w, x^{-1}]^y$  projects nontrivially to  $\mathbb{Z}_p$ , then  $(\Pi C_0^+)^{-1}(\Pi C_1^+)$  does not centralize  $a$  modulo  $\mathbb{Z}_2$ , so  $\Pi C_0^+$  and  $\Pi C_1^+$  are not both in  $\mathbb{Z}_2$ . This implies that  $\Pi C_k^+$  generates  $G'$  for some  $k$ , so the Factor Group Lemma (2.8) applies. Therefore (after replacing  $x^{-1}$  with  $x$  for simplicity), we may assume

$$[w, x]^y [x, y] \in \mathbb{Z}_2 \text{ for all } w, x, y \in S^{\pm 1} \setminus \{a\} \text{ that satisfy (9.5A) and (9.5B)}. \quad (9.5C)$$

We will show that this leads to a contradiction.

Assume, for the moment, that  $b$  centralizes  $G'$ . Then  $n = |\bar{b}| > 2$  (because Corollary 3.7 implies that  $|b| \neq 2$ ), so  $|\overline{G}| = 2n\ell > 2 \cdot 2 \cdot 4 = 16$ . Therefore (9.5B) is automatically satisfied. Let  $x, y \in S_0 \setminus \{b\}$ , such that  $x \neq y$ . We see from Note 9.3 that (9.5A) is satisfied for  $w = b^{\pm 1}$ , so (9.5C) tells us

$$[b, x]^y [x, y] \text{ and } [b^{-1}, x]^y [x, y] \text{ are both in } \mathbb{Z}_2.$$

However, we also know that  $[b^{-1}, x] = [b, x]^{-1}$  (because we are assuming in this paragraph that  $b$  centralizes  $G'$ ). Therefore

$$[b, x]^y \equiv [x, y]^{-1} \equiv [b^{-1}, x]^y = ([b, x]^{-1})^y \pmod{\mathbb{Z}_2},$$

so  $[b, x] \in \mathbb{Z}_2$  (for all  $x \in S_0$ ). Then, since  $[b, x]^y [x, y] \in \mathbb{Z}_2$ , we conclude that  $[x, y] \in \mathbb{Z}_2$ , for all  $x, y \in S_0$ . This contradicts the assumption of this subcase.

Now assume  $b$  does not centralize  $G'$ . We may assume Case 9.4 does not apply, so  $G'$  is centralized by some  $t \in S$  (and  $t \neq b$ ). Let  $w, x \in S_0 \setminus \{t\}$  with  $w \neq x$ . Combining the irredundance of  $S$  with the fact that  $t \neq b$  implies that (9.5A) is satisfied for  $y = t^{\pm 1}$  [note A.85] (unless  $\bar{w} = \bar{x}$ , when Case 4.1 applies). We may assume  $x \neq s$  (by interchanging  $w$  and  $x$ , if



necessary), so (9.5B) is satisfied. Then (9.5C) tells us

$$[w, x]^t [x, t] \text{ and } [w, x]^{t^{-1}} [x, t^{-1}] \text{ are both in } \mathbb{Z}_2.$$

Since  $t$  centralizes  $G'$ , this implies  $[x, t] \equiv [x, t^{-1}] = [x, t]^{-1} \pmod{\mathbb{Z}_2}$ , so  $[x, t] \in \mathbb{Z}_2$  (for all  $x \in S_0$ ). Since  $[w, x]^t [x, t] \in \mathbb{Z}_2$ , this implies  $[w, x] \in \mathbb{Z}_2$  (for all  $w, x \in S_0$ ). This contradicts the assumption of this subcase.  $\square$

**Case 9.6.** Assume  $b$  centralizes  $G'$ .

**Proof.** We consider two subcases.

**Subcase 9.6.1.** Assume there exists  $c \in S$ , such that  $[c, b] \notin \mathbb{Z}_p$ . We use some of the arguments of Case 8.5. We may assume  $[a, s] \in \mathbb{Z}_p$  for all  $s \in S$ . (Otherwise, Case 9.5 applies, because  $b$  centralizes  $G'$ .) Therefore  $c \neq a$ . Let  $L = (s_i)_{i=1}^r$  be a hamiltonian path from  $\bar{e}$  to  $\bar{b}$  in  $\text{Cay}(\bar{G}/\langle \bar{a} \rangle; S \setminus \{a\})$ , such that  $s_1 = c = s_r^{-1}$ , and  $L$  contains a path of the form  $[\overline{gc^\varepsilon}](c^{-\varepsilon}, b^\delta, c^\varepsilon)$  (for some  $\delta, \varepsilon \in \{\pm 1\}$ ) that is vertex-disjoint from  $\{\bar{e}, \bar{c}, \bar{b}, \bar{bc}\}$ . Now let  $C = (L, a, L^{-1}, a)$ . Then  $C$  contains vertex-disjoint paths of the form

$$[\bar{b}](a), \quad [\bar{ca}](c^{-1}, a, c), \quad [\overline{gc^\varepsilon}](c^{-\varepsilon}, b^\delta, c^\varepsilon), \quad \text{and} \quad [\overline{gab^\delta}](b^{-\delta}).$$

- Applying Lemma 2.12 to  $[\bar{b}](a)$  and  $[\bar{ca}](c^{-1}, a, c)$  (so  $s = c^{-1}$ ,  $t = a$ ,  $u = b$ , and  $h = ab$ ) will be called the “ $a$ -transform.” It multiplies the voltage by

$$\gamma_a = [b, a][a, c^{-1}].$$

- Applying Lemma 2.12 to  $[\overline{gc^\varepsilon}](c^{-\varepsilon}, b^\delta, c^\varepsilon)$  and  $[\overline{gab^\delta}](b^{-\delta})$  (so  $s = c^{-\varepsilon}$ ,  $t = b^\delta$ ,  $u = a$ , and  $h = ga$ ) will be called the “ $b$ -transform.” It multiplies the voltage by a conjugate of

$$\gamma_b = [a, b][c^{-\varepsilon}, b].$$

Since  $[a, b], [a, c] \in \mathbb{Z}_p$  and  $[b, c] \notin \mathbb{Z}_p$  we know  $\gamma_a \in \mathbb{Z}_p$  and  $\gamma_b \notin \mathbb{Z}_p$ . Also, we may also assume  $\gamma_a$  is nontrivial (by replacing  $b$  with  $b^{-1}$  if necessary). Therefore, the argument of Subcase 8.5.1 applies. Namely, choose  $C'$  to be either  $C$  or the  $b$ -transform of  $C$ , such that  $\Pi C'$  projects nontrivially to  $\mathbb{Z}_2$ . Then choose  $C''$  to be either  $C'$  or the  $a$ -transform of  $C'$ , such that  $\Pi C''$  generates  $G'$ , so the Factor Group Lemma (2.8) applies.

**Subcase 9.6.2.** Assume  $[c, b] \in \mathbb{Z}_p$  for all  $c \in S$ . Choose  $c, d \in S$ , such that  $[c, d] \notin \mathbb{Z}_p$ . Assuming that Case 9.5 and Subcase 9.6.1 do not apply, we have

$$[s, t] \in \mathbb{Z}_p \text{ for all } s \in \{a, b\} \text{ and } t \in S.$$

Therefore,  $c, d \notin \{a, b\}$ , and the element  $\gamma = [a, b][d^{-1}, a]$  is in  $\mathbb{Z}_p$ , and we may assume (by replacing  $b$  with its inverse, if necessary) that  $\gamma$  generates  $\mathbb{Z}_p$ .

Let  $S_0 = \{a, b, d\}$ , and choose a hamiltonian cycle  $C_0$  in  $\text{Cay}(\langle \bar{S}_0 \rangle; S_0)$  that contains the oriented paths  $[\bar{d}](d^{-1}, a, d)$  and  $[\bar{ab}](a)$ , and has at least two edges labelled  $x^{\pm 1}$ , for every  $x \in S_0$ . Lemma 2.12 (with  $s = d^{-1}$ ,  $t = a$ ,  $u = b$ , and  $h = b$ ) provides a hamiltonian cycle  $C_1$ , such that  $(\Pi C_0)^{-1}(\Pi C_1)$  is conjugate to  $\gamma$ , and therefore generates  $\mathbb{Z}_p$ . Furthermore,  $C_1$  contains all of the oriented edges of  $C_0$  that are not in these two above-mentioned paths, so Lemma 3.15(2) applies (with  $g = b$  and  $t = d$ ).  $\square$

**Case 9.7.** Assume that none of the preceding cases apply.

**Proof.** This implies that:

- #1.  $[a, b] \in \mathbb{Z}_p$ . (Otherwise, Case 9.5 applies.)

- #2. If  $s \in S$ , and there exists  $t \in S$ , such that  $t$  inverts  $G'$  and  $\mathbb{Z}_p \subseteq \langle [s, t] \rangle$ , then  $s$  inverts  $G'$ . (If  $s$  does not invert  $G'$ , then we see from Assumption 9.1 that  $s$  centralizes  $G'$ , so Case 9.6 applies with  $s$  and  $t$  in the roles of  $b$  and  $a$ , respectively.)
- #3. There exists  $c \in S$ , such that  $c$  centralizes  $G'$ . (Otherwise, Case 9.4 applies.) From (#2), we know  $[a, c] \in \mathbb{Z}_2$ .

**Subcase 9.7.1.** Assume  $\langle [s, c] \rangle \neq \mathbb{Z}_2$ , for some  $s \in S \setminus \{c\}$ . Suppose, for the moment, that  $s$  centralizes  $G'$ . Then Lemma 3.6 implies  $[a, [s, c]] = [[a, s], [a, c]] = e$  (because  $G'$  is abelian), so  $[s, c]$  projects trivially to  $\mathbb{Z}_p$ . Since  $\langle [s, c] \rangle \neq \mathbb{Z}_2$ , we conclude from this that  $[s, c] = e$ , so Lemma 2.16 applies.

We may now assume  $s$  does not centralize  $G'$ , so there is no harm in assuming that  $s = a$ . Since (#2) implies that  $[a, c] \in \mathbb{Z}_2$ , we see that  $[a, c]$  must be trivial. Let  $H = \langle S \setminus \{c\} \rangle$ . We may assume  $\mathbb{Z}_2 \not\subseteq H$ , for otherwise  $H \triangleleft G$ , so Lemma 2.13 applies with  $s = c$  and  $t = a$ . Therefore,  $[x, y] \in \mathbb{Z}_p$  for all  $x, y \in S \setminus \{c\}$ , but there is some  $d \in S \setminus \{c\}$ , such that  $[c, d]$  projects nontrivially to  $\mathbb{Z}_2$ .

Similarly, we may assume  $\mathbb{Z}_p \not\subseteq \langle S \setminus \{a\} \rangle$ , for otherwise  $\langle S \setminus \{a\} \rangle \triangleleft G$ , so Lemma 2.13 applies with  $s = a$  and  $t = c$ . This means  $[x, y] \in \mathbb{Z}_2$  for all  $x, y \in S \setminus \{a\}$ . In particular, since  $b$  and  $d$  are in both  $S \setminus \{a\}$  and  $S \setminus \{c\}$ , we must have  $[b, d] \in \mathbb{Z}_2 \cap \mathbb{Z}_p = \{e\}$ .

Choose a hamiltonian cycle  $C_0$  in  $\text{Cay}(\overline{H}; S \setminus \{c\})$  that contains the oriented paths  $[\overline{d}](d^{-1}, b, d)$  and  $[\overline{ab}](b)$ . If we apply Lemma 2.12 to these paths (so  $s = d^{-1}$ ,  $t = b$ , [note A.91]  $u = a$ , and  $h = a$ ), then the voltage is multiplied by a conjugate of  $[b, a][b, d^{-1}]$ , which is a [note A.92] generator of  $\mathbb{Z}_p$  (since  $[a, b]$  generates  $\mathbb{Z}_p$  and  $[b, d]$  is trivial). Therefore, Lemma 3.15(1) applies with  $s = t = d$  and  $u = a$ .

**Subcase 9.7.2.** Assume  $\langle [s, c] \rangle = \mathbb{Z}_2$ , for all  $s \in S \setminus \{c\}$ . For convenience, let  $\widehat{G} = G/\mathbb{Z}_2$  and  $\widehat{H} = \langle \widehat{S} \setminus \{\widehat{c}\} \rangle$ . Then  $|\widehat{H}'| = p$  is prime, so Theorem 1.1 provides a hamiltonian path  $L$  in  $\text{Cay}(\widehat{H}; S \setminus \{c\})$ . Since  $\widehat{c}$  is central in  $\widehat{G}$ , there is a spanning subgraph of  $\text{Cay}(\widehat{G}; S)$  that is isomorphic to the Cartesian product  $L \square (\widehat{c}^{\ell-1})$ , where  $\ell = |\widehat{G} : \langle S \setminus \{c\} \rangle|$ . Since  $|\widehat{G}|$  is even, it is easy to find a hamiltonian cycle  $C$  in  $L \square (\widehat{c}^{\ell-1})$  (see Lemma 2.10), and this yields a hamiltonian cycle  $\widehat{C}$  in  $\text{Cay}(\widehat{G}; S)$ .

To complete the proof, we carry out a straightforward (and well-known) calculation to verify that  $\Pi \widehat{C}$  is nontrivial, so the Factor Group Lemma (2.8) applies.

If we view the Cartesian product  $L \square (\widehat{c}^{\ell-1})$  as a grid of squares, then the interior of the hamiltonian cycle  $C$  is a union of squares of the grid. Graph theoretically, this means  $C$  is the connected sum of some number  $N$  of digons of the form  $[g](t, t^{-1})$  (where  $t \in S^{\pm 1}$ ). Note that if  $\mathcal{C}$  is an  $r$ -cycle (with  $r \geq 2$ ), then  $\mathcal{C} \#_t^s(t, t^{-1})$  is an  $(r+2)$ -cycle. Therefore, since the length of  $C$  is  $|\widehat{G}|$ , we have  $2N = |\widehat{G}| \equiv 0 \pmod{4}$ , so  $N$  is even.

Now, each 4-cycle in  $L \square (\widehat{c}^{\ell-1})$  is of the form  $[\widehat{g}](s^{-1}, t^{-1}, s, t)$ , where one of  $s$  and  $t$  is in  $\{c^{\pm 1}\}$ , and the other is in  $S^{\pm 1} \setminus \{c^{\pm 1}\}$ . This means that in any connected sum  $\mathcal{C} \#_t^s[g](t, t^{-1})$ , one of  $s$  and  $t$  is in  $\{c^{\pm 1}\}$ , and the other is in  $S^{\pm 1} \setminus \{c^{\pm 1}\}$ . By the assumption of this subcase, we conclude that  $[s, t] = z$ , where  $z$  is the generator of  $\mathbb{Z}_2$ . Therefore

$$\begin{aligned}
 \Pi C &= \Pi \left( [\widehat{g}_1](t_1, t_1^{-1}) \#_{t_2}^{s_2} [\widehat{g}_2](t_2, t_2^{-1}) \#_{t_3}^{s_3} \cdots \#_{t_N}^{s_N} [\widehat{g}_N](t_N, t_N^{-1}) \right) \\
 &= \prod_{i=2}^N [s_i, t_i] \quad \left( \begin{array}{l} \text{Corollary 3.14} \\ \text{and } \Pi(t, t^{-1}) = e \end{array} \right) \\
 &= z^{N-1} \\
 &\neq e \quad (\text{mod } \mathbb{Z}_p) \quad (N-1 \text{ is odd}). \quad \square
 \end{aligned}$$

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## A Notes to aid the referee

**A.1.** Write  $C = [Nv](s_i)_{i=1}^n$ .

(1) Suppose  $C$  has another representation:  $C = [Nw](t_j)_{j=1}^n$ . Since  $Nw$  is a vertex on  $C$ , there is some  $\ell$ , such that  $Nvs_1s_2 \cdots s_\ell = Nw$ . Then  $t_j = s_{j-\ell}$  for all  $j$  (with subscripts read modulo  $n$ ). Also, letting  $g = (s_1s_2 \cdots s_\ell)^{-1}$ , we have  $Nwg = Nv$ , so (since  $N$  is normal) there is some  $h \in N$ , such that  $wg = vh$ . Therefore

$$\begin{aligned}
 {}^w\left(\prod_{j=1}^n t_j\right) &= {}^w\left(\left(\prod_{i=\ell+1}^n s_i\right)\left(\prod_{i=1}^\ell s_i\right)\right) \\
 &= {}^w\left(g\left(\prod_{i=1}^\ell s_i\right)\left(\prod_{i=\ell+1}^n s_i\right)g^{-1}\right) \\
 &= {}^w\left(g\left(\prod_{i=1}^n s_i\right)g^{-1}\right) \\
 &= {}^{wg}\left(\prod_{i=1}^n s_i\right) \\
 &= {}^{vh}\left(\prod_{i=1}^n s_i\right) \\
 &= {}^v\left(\prod_{i=1}^n s_i\right) \quad \left(\begin{array}{l} \text{since } N \text{ is abelian, we have} \\ {}^h x = x \text{ for all } x \in N \end{array}\right).
 \end{aligned}$$

This means that the two representations  $[Nw](t_j)_{j=1}^n$  and  $[Nv](s_i)_{i=1}^n$  yield the same value for the voltage, so the voltage is well defined.

(2) We have  $gC = [Ngv](s_i)_{i=1}^n$ , so

$$\Pi gC = {}^{gv}\left(\prod_{i=1}^n s_i\right) = {}^g\left({}^v\left(\prod_{i=1}^n s_i\right)\right) = {}^g(\Pi C).$$

(3) We have  $-C = [Nv](s_n^{-1}, s_{n-1}^{-1}, \dots, s_1^{-1})$ , so

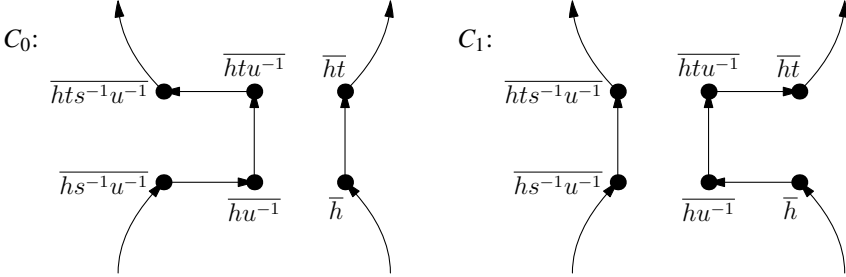
$$\Pi(-C) = {}^v(s_n^{-1}s_{n-1}^{-1} \cdots s_1^{-1}) = {}^v((s_1s_2 \cdots s_n)^{-1}) = ({}^v(s_1s_2 \cdots s_n))^{-1} = (\Pi C)^{-1}.$$

**A.2.** We have

$$\begin{aligned}
 \Pi(s_i)_{i=1}^r &= \prod_{i=1}^{k\ell/2} ((\Pi L)t_{2i-1}(\Pi L)^{-1}t_{2i})t_{k\ell}^{-1} \\
 &= \prod_{i=1}^{k\ell/2} (t_{2i-1}t_{2i})t_{k\ell}^{-1} \quad (H \text{ is abelian}) \\
 &= \prod_{i=1}^{k\ell-1} t_i \\
 &= x^p y^q.
 \end{aligned}$$

**A.3. Case 1.** Assume that  $C_0$  contains  $[\bar{h}](t)$ . Construct  $C_1$  by replacing:

- the oriented edge  $[\bar{h}](t)$  with the oriented path  $[\bar{h}](u^{-1}, t, u)$ , and
- the oriented path  $[\overline{hs^{-1}u^{-1}}](s, t, s^{-1})$  with the oriented edge  $[\overline{hs^{-1}u^{-1}}](t)$ .



To calculate the voltage of  $C_1$ , write  $C_0 = [\bar{h}](s_1, \dots, s_n)$ . Then  $s_1 = t$  and there is some  $\ell$  with  $\overline{s_1} \cdots \overline{s_\ell} = \overline{u^{-1}}$ , so  $(s_\ell, s_{\ell+1}, s_{\ell+2}) = (s, t, s^{-1})$ , and we have

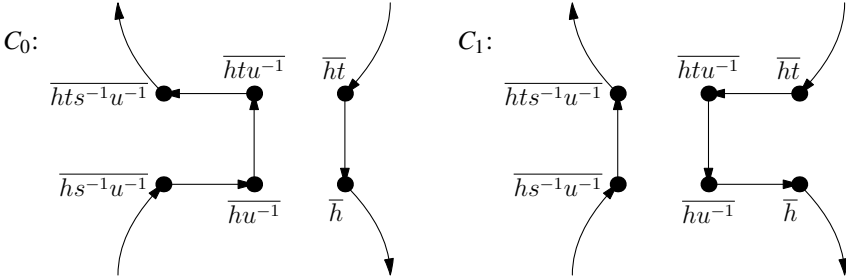
$$C_1 = [\bar{h}](u^{-1}, t, u, (s_i)_{i=2}^{\ell-1}, t, (s_i)_{i=\ell+3}^n).$$

Note that if we let  $\pi = \prod_{i=1}^\ell s_i$ , then  $\pi \equiv u^{-1} \pmod{N}$ , so  $\pi x = x^u$  for all  $x \in N$  (since  $N$  is commutative). Therefore

$$\begin{aligned} (\Pi C_1)^h &= (u^{-1}tu) \left( \prod_{i=2}^{\ell-1} s_i \right) t \left( \prod_{i=\ell+3}^n s_i \right) \\ &= (u^{-1}tu) t^{-1} \left( \prod_{i=1}^\ell s_i \right) s^{-1} t s t^{-1} \left( \prod_{i=\ell+1}^n s_i \right) \\ &= [u, t^{-1}] \left( \prod_{i=1}^\ell s_i \right) [s, t^{-1}] \left( \prod_{i=\ell+1}^n s_i \right) \\ &= [u, t^{-1}]^\pi [s, t^{-1}] \left( \prod_{i=1}^\ell s_i \right) \left( \prod_{i=\ell+1}^n s_i \right) \\ &= [u, t^{-1}] [s, t^{-1}]^u (\Pi C_0)^h. \end{aligned}$$

**Case 2.** Assume that  $C_0$  contains  $[\overline{ht}](t^{-1})$ . Construct  $C_1$  by replacing:

- the oriented edge  $[\overline{ht}](t^{-1})$  with the oriented path  $[\overline{ht}](u^{-1}, t^{-1}, u)$ , and
- the oriented path  $[\overline{hs^{-1}u^{-1}}](s, t, s^{-1})$  with the oriented edge  $[\overline{hs^{-1}u^{-1}}](t)$ .



To calculate the voltage of  $C_1$ , write  $C_0 = [\overline{ht}](s_1, \dots, s_n)$ . Then  $s_1 = t^{-1}$  and there is some  $\ell$  with  $\overline{s_1} \cdots \overline{s_\ell} = \overline{t^{-1}u^{-1}}$ , so  $(s_\ell, s_{\ell+1}, s_{\ell+2}) = (s, t, s^{-1})$ , and we have

$$C_1 = [\overline{ht}](u^{-1}, t^{-1}, u, (s_i)_{i=2}^{\ell-1}, t, (s_i)_{i=\ell+3}^n).$$

Then

$$\begin{aligned}
 (\Pi C_1)^{ht} &= (u^{-1}t^{-1}u) \left( \prod_{i=2}^{\ell-1} s_i \right) t \left( \prod_{i=\ell+3}^n s_i \right) \\
 &= (u^{-1}t^{-1}u) t \left( \prod_{i=1}^{\ell} s_i \right) s^{-1} t s t^{-1} \left( \prod_{i=\ell+1}^n s_i \right) \\
 &= [u, t] \left( \prod_{i=1}^{\ell} s_i \right) [s, t^{-1}] \left( \prod_{i=\ell+1}^n s_i \right) \\
 &= [u, t] \pi[s, t^{-1}] \left( \prod_{i=1}^{\ell} s_i \right) \left( \prod_{i=\ell+1}^n s_i \right) \\
 &= [u, t] [s, t^{-1}]^{tu} (\Pi C_0)^{ht}.
 \end{aligned}$$

Conjugating both sides by  $t^{-1}$  yields

$$(\Pi C_1)^h = [u, t]^{t^{-1}} [s, t^{-1}]^u (\Pi C_0)^h.$$

Now note that

$$[u, t]^{t^{-1}} = t(u^{-1}t^{-1}ut)t^{-1} = tu^{-1}t^{-1}u = [t^{-1}, u].$$

**A.4.** Case 4.5 of [11] (on page 95) considers certain groups of order 27. Near the start of [11, §4] (on page 92), it is stated that “In every case except 4.5, we use the Factor Group Lemma 2.3 on  $G/G'$ ” Replacing  $G$  with  $\widehat{G}$ , this means there is a hamiltonian cycle in  $\text{Cay}(\widehat{G}/\widehat{G}'; S)$  whose voltage generates  $\widehat{G}'$  (unless  $|\widehat{G}| = 27$ , which we have ruled out).

**A.5.** Fix some  $\widehat{g} \in \widehat{G} \setminus Z(\widehat{G})$ , and define  $\varphi: \widehat{G} \rightarrow \widehat{G}'$  by  $\varphi(x) = [x, \widehat{g}]$ . From Lemma 3.6, we know that  $\varphi$  is a homomorphism. Since  $\widehat{g} \notin Z(\widehat{G})$ , this homomorphism is nontrivial, so it must be surjective (since  $|\widehat{G}'| = q$  is prime). Therefore  $|\widehat{G} : \ker \varphi| = |\widehat{G}'| = q$ . Also, we have  $\widehat{G}' \subseteq Z(\widehat{G}) \subseteq \ker \varphi$ . So  $|\widehat{G} : \widehat{G}'|$  is divisible by  $q$ .

**A.6.** If  $T$  is a subset of  $S$ , then it is obvious that  $\text{Cay}(G; T)$  is a subgraph of  $\text{Cay}(G; S)$ . Therefore, in order to show that every connected Cayley graph on  $G$  has a hamiltonian cycle, it suffices to consider only the irredundant generating sets.

**A.7.** Suppose  $\mathbb{Z}_p \cap Z(G)$  is nontrivial. Since  $\mathbb{Z}_p$  has prime order, this implies  $\mathbb{Z}_p \subseteq Z(G)$ . However,  $\mathbb{Z}_2$  is a normal subgroup that has no automorphisms, so  $\mathbb{Z}_2 \subseteq Z(G)$ . Therefore,  $Z(G)$  contains both  $\mathbb{Z}_2$  and  $\mathbb{Z}_p$ , and therefore contains all of  $G'$ . This contradicts the fact that  $G$  is not nilpotent.

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**A.8.** If  $\mathbb{Z}_p \not\subseteq \langle [a, b] \rangle$ , then, since  $[a, b] \in G' = \mathbb{Z}_2 \times \mathbb{Z}_p$ , we must have  $[a, b] \in \mathbb{Z}_2$ . If this is true for all  $b \in S$ , then  $[a, g] \in \mathbb{Z}_2$  for all  $g \in G$  (because  $\langle S \rangle = G$  and  $\mathbb{Z}_2 \triangleleft G$ ). In particular,  $[a, \mathbb{Z}_p] \subseteq \mathbb{Z}_2$ . However, we also have  $[a, \mathbb{Z}_p] \subseteq \mathbb{Z}_p$ , because  $\mathbb{Z}_p \triangleleft G$ . Therefore,  $[a, \mathbb{Z}_p] \subseteq \mathbb{Z}_2 \cap \mathbb{Z}_p = \{e\}$ . This contradicts (3.3A).

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**A.9.** We have

$$xy[xy, z] = (xy)^z = x^z y^z = x[x, z] \cdot y[y, z] = xy[x, z][y, z].$$


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**A.10.** Let  $\widehat{G} = G/\mathbb{Z}_2$ . We have  $[\widehat{x}, \widehat{y}^p] = [\widehat{x}, \widehat{y}]^p = \widehat{e}$ , so  $\widehat{y}^p \in Z(\widehat{G})$ . If  $p \nmid |y|$ , this implies  $\widehat{y} \in Z(\widehat{G})$ , which contradicts the fact that  $[\widehat{x}, \widehat{y}]$  is nontrivial (because  $\mathbb{Z}_p \subseteq \langle [x, y] \rangle$ ).

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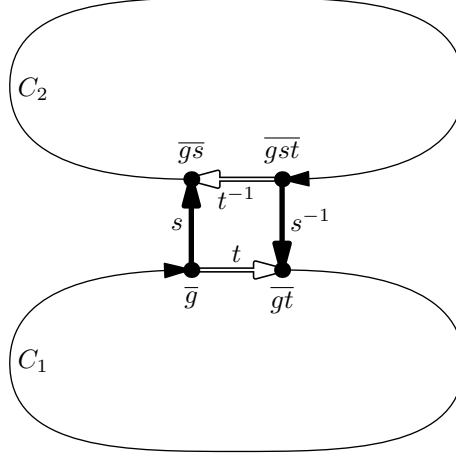
**A.11.** Let  $\widehat{G} = G/\mathbb{Z}_p$ . By assumption, there exists  $s \in S_0$ , such that  $\langle [\widehat{g}, \widehat{s}] \rangle = \widehat{\mathbb{Z}}_2 = \widehat{G}'$ . Every element of  $\widehat{G}$  centralizes  $\mathbb{Z}_2 = \widehat{G}'$ , so Lemma 3.6 tells us that the map  $\varphi(x) = [x, s]$  is a homomorphism from  $\langle \widehat{g}, \widehat{S}_0 \rangle$  to  $\mathbb{Z}_2$ . Since  $\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$  (and  $s \in S_0$ ), we know  $\widehat{S}_0$  is contained in the kernel of  $\varphi$ . But  $\langle \varphi(\widehat{g}) \rangle = \mathbb{Z}_2$ , so the kernel of  $\varphi$  is a subgroup of index 2 in  $\langle \widehat{g}, \widehat{S}_0 \rangle$ . Therefore

$$\frac{|\langle \overline{g}, \overline{S_0} \rangle|}{|\langle \overline{S_0} \rangle|} = \frac{|\langle \overline{g}, \overline{S_0} \rangle|}{|\ker \varphi|} \cdot \frac{|\ker \varphi|}{|\langle \overline{S_0} \rangle|} = 2 \cdot \frac{|\ker \varphi|}{|\langle \overline{S_0} \rangle|} \quad \text{is even.}$$


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**A.12.** The connected sum  $C_1 \#_t^s C_2$  joins  $C_1$  and  $C_2$  into a single large cycle by replacing the two white edges labelled  $t$  and  $t^{-1}$  with the two black edges labelled  $s$  and  $s^{-1}$ .



**A.13.** Let  $S_0^* = S_0 \cup \{t\}$ . We verify the hypotheses of Lemma 3.15 with  $S_0^*$  in the role of  $S_0$  and with  $t$  in the role of  $s$ .

- $(\Pi C_0)^{-1}(\Pi C_1)$  is a nontrivial element of  $\mathbb{Z}_p$  (by assumption).
- By construction of the connected sum,  $C'_0$  and  $C'_1$  both contain the oriented edge  $[\bar{g}](t)$ .
- By construction of the connected sum (and the fact that  $s_1 = s$ ),  $C'_0$  contains the oriented edges  $[\bar{g}](t)$  and  $[\bar{gs}](t^{-1})$ . Also, for every  $x \in S_0$ ,  $C_0$  contains at least two edges  $[\bar{v}](x^{\pm 1})$  and  $[\bar{w}](x^{\pm 1})$  that are labelled either  $x$  or  $x^{-1}$ . Then the subgraph of  $C'_0$  induced by  $C_0$  contains at least one of these edges, and the subgraph of  $C'_0$  induced by  $t^{n-1}C_0$  contains either  $[\bar{t^{n-1}v}](x^{\pm 1})$  or  $[\bar{t^{n-1}w}](x^{\pm 1})$ ; so  $C'_0$  contains at least two edges that are labelled either  $x$  or  $x^{-1}$ .
- We know  $c \notin S_0$  and  $\mathbb{Z}_2 \subseteq \langle [c, t] \rangle$ . The latter implies  $c \neq t$ , so  $c \notin S_0 \cup \{t\} = S_0^*$ .
- We are letting  $s = t$ .
- By assumption, either
  1. there exists  $u \in S \setminus \{c\}$ , such that  $\mathbb{Z}_2 \not\subseteq \langle [u, c] \rangle$ , or
  2.  $|\bar{G} : \langle \bar{S}_0, \bar{t} \rangle|$  is even.

The first condition makes no mention of  $S_0$ ,  $s$ , or  $t$ , so remains true with  $S_0^*$  in the role of  $S_0$  and with  $t$  in the role of  $s$ . Since  $S_0^* = S_0 \cup \{t\}$ , we have  $S_0^* \cup \{t\} = S_0 \cup \{t\}$ . So the second condition tells us that  $|\bar{G} : \langle \bar{S}_0^*, \bar{t} \rangle|$  is even.

**A.14.** Suppose  $|\overline{G} : \langle \overline{S_0} \rangle| = 2$ , so  $C = C_0 \#_{t_1}^c - cC_0$ . Then, calculating mod  $\mathbb{Z}_p$ , we have

$$\begin{aligned}
 0 &\neq \Pi C \\
 &\equiv \Pi C_0 \cdot \Pi(-cC_0) \cdot [c, t_1] && \text{(Corollary 3.14)} \\
 &\equiv \Pi C_0 \cdot \Pi C_0 \cdot [c, t_1] && \text{(Lemma 2.6(2))} \\
 &\equiv [c, t_1] && (x^2 \in \mathbb{Z}_p \text{ for all } x \in G', \text{ since } G' = \mathbb{Z}_2 \times \mathbb{Z}_p).
 \end{aligned}$$

By the definition of  $u$ , this implies  $u \neq t_1$ . So  $t_1 = t$ .

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**A.15.** The choice of the oriented edge  $[\overline{g_i}](t_i)$  of  $C_0$  that is used in the connected sum is arbitrary, except that  $t_1$  was chosen to make the projection of  $\Pi C$  to  $\mathbb{Z}_2$  is nontrivial. Therefore, if  $n > 1$ , then we may use any edge that we want in order to make the connected sum  $(-1)^{n-1} \pi_{n-1} C_0 \#_s^{s_n} (-1)^n \pi_n C_0$ .

So we may now assume  $n = 1$ . This means  $|\overline{G} : \langle \overline{S_0} \rangle| = 2$ . Therefore, by assumption, we must have  $s = t$ . Also, as was mentioned in the proof, we must have  $t_1 = t$ . So  $t_1 = s$ . Therefore, we may assume that the connected sum  $C_0 \#_{t_1}^c - cC_0$  is relative to the oriented edge  $[\overline{g_c}](s)$  of  $cC_0$  that is also in  $cC_1$ .

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**A.16.** Suppose there exist  $d \in S \setminus S_0$  and  $u \in S \setminus \{d\}$ , such that  $[u, d]$  projects trivially to  $\mathbb{Z}_2$ . Note that, by the assumption of this case, we must have  $d \neq c$ .

1. By applying the assumption of this case with  $d$  in the place of  $u$  (and noting that  $d \neq c$ ), we see that  $\mathbb{Z}_2 \subseteq [d, c]$ .
2. By the choice of  $u$ , we know that  $\mathbb{Z}_2 \not\subseteq \langle [u, d] \rangle$ .

Therefore, the hypotheses of the lemma are satisfied with  $d$  and  $c$  in the roles of  $c$  and  $t$ , respectively. Furthermore (2) tells us that Case 1 applies.

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**A.17.** Suppose  $\langle S \setminus \{t\} \rangle$  contains  $\mathbb{Z}_p$ . Note that  $\langle S \setminus \{t\} \rangle$  also contains  $s$ . Therefore, we have

$$\begin{aligned} \langle S \setminus \{t\}, \mathbb{Z}_2 \rangle &= \langle S \setminus \{t\}, s, \mathbb{Z}_p, \mathbb{Z}_2 \rangle = \langle S \setminus \{t\}, s, G' \rangle = \langle S \setminus \{t\}, s, \gamma \rangle \\ &\supseteq \langle S \setminus \{t\}, s\gamma \rangle = \langle S \setminus \{t\}, t \rangle = \langle S \rangle = G. \end{aligned}$$

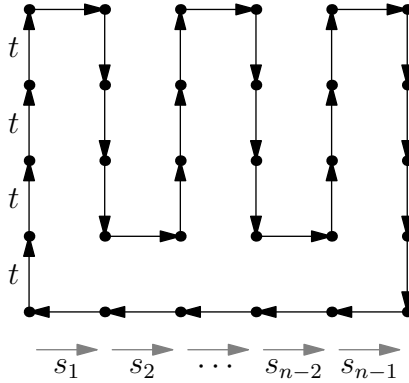
So Lemma 3.4 tells us that  $\langle S \setminus \{t\} \rangle = G$ . This contradicts the irredundance of  $S$ , so we conclude that  $\langle S \setminus \{t\} \rangle$  does not contain  $\mathbb{Z}_p$ . A similar argument shows that  $\langle S \setminus \{s\} \rangle$  does not contain  $\mathbb{Z}_p$ .

Suppose  $u$  is an element of  $S \setminus \{s, t\}$  that does not centralize  $\mathbb{Z}_p$ . Then  $u$  is not in the center of  $G/\mathbb{Z}_2$ , so there is some  $x \in S$ , such that  $[x, u] \notin \mathbb{Z}_2$ . We may assume (perhaps after interchanging  $s$  and  $t$ ) that  $x \neq s$ , so  $x \in S \setminus \{s\}$ . Then  $u$  and  $x$  are both in  $S \setminus \{s\}$ , so the commutator subgroup of  $\langle S \setminus \{s\} \rangle$  is not contained in  $\mathbb{Z}_2$ . Since  $G' = \mathbb{Z}_2 \times \mathbb{Z}_p$ , this implies that the commutator subgroup of  $\langle S \setminus \{s\} \rangle$  contains  $\mathbb{Z}_p$ . So  $\langle S \setminus \{s\} \rangle$  contains  $\mathbb{Z}_p$ . This contradicts the preceding paragraph, so we conclude that every element of  $S \setminus \{s, t\}$  that centralizes  $\mathbb{Z}_p$ .

**A.18.** Suppose  $s$  centralizes  $\mathbb{Z}_p$ . Since  $t = s\gamma$  and it is obvious that  $\gamma$  centralizes  $\mathbb{Z}_p$  (because  $\gamma \in G' = \mathbb{Z}_2 \times \mathbb{Z}_p$ ), we conclude that  $t$  also centralizes  $\mathbb{Z}_p$ . From the conclusion of the preceding paragraph, we conclude that every element of  $S$  centralizes  $\mathbb{Z}_p$ . Since  $S$  generates  $G$ , this implies that every element of  $G$  centralizes  $\mathbb{Z}_p$ . This contradicts (3.3A), so we conclude that  $s$  does not centralize  $\mathbb{Z}_p$ . A similar argument shows that  $t$  does not centralize  $\mathbb{Z}_p$ .

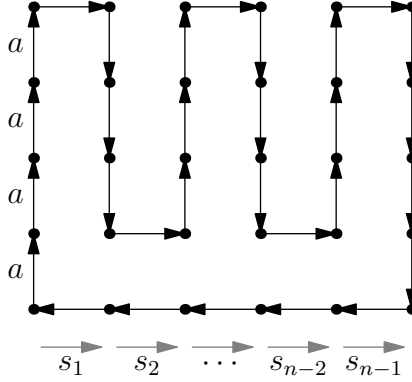
**A.19.** Since  $n$  is even, we may let

$$C = (t, (t^{m-2}, s_{2i-1}, t^{-(m-2)}, s_{2i})_{i=1}^{n/2} \#, t^{-1}, (s_{n-j}^{-1})_{j=1}^{n-1}).$$





A.23.



A.24. Since  $\widehat{G}' = \mathbb{Z}_2$ , we have  $\widehat{G}' \subseteq Z(\widehat{G})$ , which implies  $[x, yz][x, y][x, z]$  for all  $x, y, z \in \widehat{G}$ . Therefore  $[x, \widehat{a}^{m-2}] = [x, \widehat{a}]^{m-2}$ . Since  $m-2$  is even and  $[x, \widehat{a}] \in \widehat{G}' = \mathbb{Z}_2$ , this implies  $[x, \widehat{a}^{m-2}]$  is trivial. Since  $x$  is an arbitrary element of  $\widehat{G}$ , this implies  $\widehat{a} \in Z(\widehat{G})$ .

A.25. We have  $[a^\delta, b^{-1}] = [b^{-1}, a^\delta]^{-1}$ . Therefore:

- Calculating modulo  $\mathbb{Z}_p$ , we have

$$[a^\delta, b^{-1}]^{a^\delta} \equiv [a^\delta, b^{-1}] = [b^{-1}, a^\delta]^{-1},$$

since  $a$  centralizes  $\mathbb{Z}_2$ . Therefore  $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta}$  is trivial modulo  $\mathbb{Z}_p$ . In other words,  $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta} \in \mathbb{Z}_p$ .

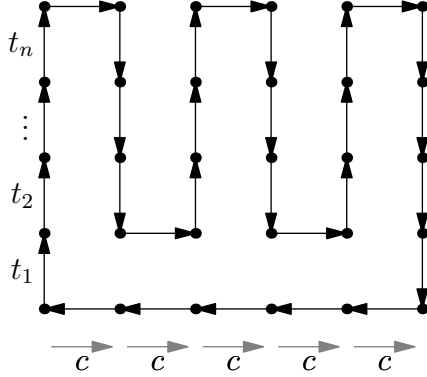
- We have  $[a^\delta, b^{-1}]^{a^\delta} \neq [b^{-1}, a^\delta]^{-1}$ , since  $a^\delta = a^{\pm 1}$  does not centralize  $\mathbb{Z}_p$ . This implies  $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta}$  is nontrivial.

Combining these two observations tells us that  $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta}$  is a generator of  $\mathbb{Z}_p$ .

A.26. By assumption,  $a$  is in the center of  $G/\mathbb{Z}_p$ , so  $\langle a, \mathbb{Z}_p \rangle \triangleleft G$ . Let  $\widehat{G} = G/\langle a, \mathbb{Z}_p \rangle$ . Since  $\mathbb{Z}_2 \subseteq \langle [\widehat{c}, \widehat{d}] \rangle$ , Corollary 3.8 tells us that  $|\widehat{c}|$  is even.

**A.27.** Let  $\hat{H} = \langle \bar{S} \setminus \{\bar{d}\} \rangle / \langle \bar{a} \rangle$ , and let  $w = |\hat{C}|$ . It has been pointed out that the image of  $c$  in  $\bar{G} / \langle \bar{a} \rangle$  has even order, which means that  $w$  is even.

Suppose  $\hat{b} \notin \langle \hat{C} \rangle$ . We may let  $(t_j)_{j=1}^\ell$  be a hamiltonian path in  $\text{Cay}(\hat{H} / \langle \hat{C} \rangle; S \setminus \{c, d\})$ , such that  $t_1 = b$ . Then we may take  $(s_i)_{i=1}^n$  to be the following hamiltonian cycle:

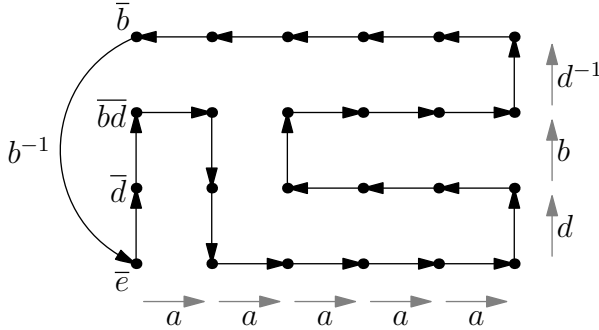


If  $\langle \hat{C} \rangle = \hat{H}$ , then we may write  $\hat{b} = \hat{C}^q$  with  $1 \leq q \leq w-1$ , so we let

$$(s_i)_{i=1}^n = (b, c^{-(q-1)}, b, c^{w-q-1}).$$

We now assume  $\hat{b} \in \langle \hat{C} \rangle \neq \hat{H}$ . If  $\hat{b} = \hat{C}$ , then we modify the above-pictured hamiltonian cycle, by replacing a single occurrence of  $c$  with  $b$ . Otherwise, we write  $\hat{b} = \hat{C}^q$  with  $1 \leq q \leq w-2$ , and replace the path  $(c^{-(w-1)})$  at the end of the above-pictured hamiltonian cycle with  $(b, c^{q-1}, b, c^{-(w-q-2)})$ .

**A.28.**



**A.29.** It is a general group-theoretic fact that if  $x$  does not invert  $[x, y]$ , then  $x^2$  does not commute with  $y$ . This is because

$$y^{x^2} = (y^x)^x = (y[y, x])^x = y^x [y, x]^x = y [y, x] [y, x]^x,$$

so  $y^{x^2} = y$  iff  $[y, x]^x = [y, x]^{-1}$ .

Let  $\widehat{G} = G/\mathbb{Z}_2$ . Since  $|\widehat{a}| = 3$  (and  $a$  does not centralize  $G'$ ), we know that  $a$  acts on  $\mathbb{Z}_p$  via an automorphism  $\phi$  of order 3. So  $\widehat{a}$  does not invert  $[\widehat{a}, \widehat{b}]$  (because  $[\widehat{a}, \widehat{b}]$  is a generator of  $\mathbb{Z}_p$ ). Therefore, the general fact tells us that  $\widehat{a}^2$  does not commute with  $\widehat{b}$ , so  $[\widehat{a}^2, \widehat{b}]$  is a generator of  $\mathbb{Z}_p$ . Since  $a^2$  acts on  $\mathbb{Z}_p$  via  $\phi^2$ , which is an automorphism of order 3, we know that  $\widehat{a}^2$  does not invert  $[\widehat{a}^2, \widehat{b}]$ . So the general fact tells us that  $(\widehat{a}^2)^2$  does not commute with  $\widehat{b}$ . This means  $b^{(a^2)^2} \not\equiv b \pmod{\mathbb{Z}_2}$ . Conjugating by  $a^{-2}$ , we conclude that  $b^{a^2} \not\equiv b^{a^{-2}} \pmod{\mathbb{Z}_2}$ .

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**A.30.** Suppose  $[b^{a^2}, d]$  and  $[b^{a^{-2}}, d]$  are both in  $\mathbb{Z}_2$ . This means that  $b^{a^2}$  and  $b^{a^{-2}}$  both commute with  $d \pmod{\mathbb{Z}_2}$ , so the product  $b^{a^2}(b^{a^{-2}})^{-1}$  also commutes with  $d \pmod{\mathbb{Z}_2}$ . For convenience, call this product  $\gamma$ . Then

$$\gamma = b^{a^2}(b^{a^{-2}})^{-1} \equiv b \cdot b^{-1} = e \pmod{G'},$$

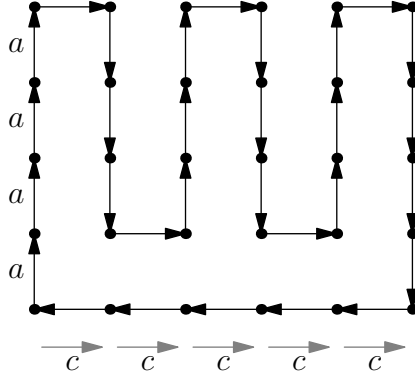
so  $\gamma \in G'$ . Furthermore, we have observed that  $b^{a^2} \not\equiv b^{a^{-2}} \pmod{\mathbb{Z}_2}$ , so  $\gamma \notin \mathbb{Z}_2$ . Therefore  $\mathbb{Z}_p \subseteq \langle \gamma \rangle$ . Since  $\gamma$  commutes with  $d \pmod{\mathbb{Z}_2}$ , we conclude that  $d$  centralizes  $\mathbb{Z}_p$ , and therefore centralizes  $G'$ . This is a contradiction.

---

**A.31.** If  $b$  centralizes  $G'$ , then  $\mathbb{Z}_p$  is in the center of  $\langle b, d \rangle$ . Therefore, by using Lemma 3.6 as in the proof of Corollary 3.9 (but replacing 2 with  $p$ ), we see that  $|\langle \overline{b}, d \rangle|$  is divisible by  $p^2$ . Since  $p \neq 2$  and  $|\overline{G}| = 12$ , this is a contradiction.

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A.32.

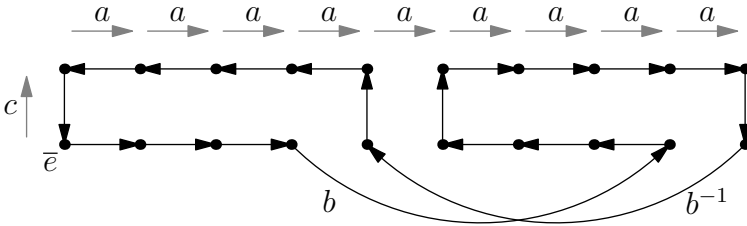


**A.33.** To see that  $(\Pi C_0)^{-1}(\Pi C_1)$  is a generator of  $\mathbb{Z}_p$ , we repeat the calculation in the last paragraph of [4, p. 60] (using our notation). However, it is important to note that  $a^k$  centralizes  $\gamma$  (because  $a$  inverts  $G'$  and  $k$  is even). We have:

$$\begin{aligned}
 (\Pi C_0)(\Pi C_1)^{-1} &= (ba^{-(k-4)}ba^{m-2k-2}ba^{-1}ba^2b^{-2}a^{k-3})^{-1}(a^m)^{-1} \\
 &= (ba^{-k})a^4(ba^{-k})a^{m-2}(a^{-k}b)a^{-1}(ba^2b^{-1})(b^{-1}a^k)a^{-3}a^{-m} \\
 &= \gamma a^4 \gamma a^{m-2} \gamma a^{-1} a^2 \gamma^{-1} a^{-3} a^{-m} \\
 &= \gamma^4 \quad (\text{since } a \text{ inverts } \gamma \text{ and } m \text{ is even}).
 \end{aligned}$$

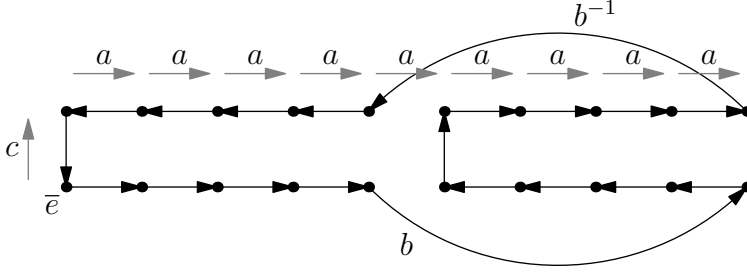
This is a generator of  $\mathbb{Z}_p$ . Therefore,  $(\Pi C_0)^{-1}(\Pi C_1)$  is also a nontrivial element of  $\mathbb{Z}_p$ , since it is conjugate to the inverse of  $(\Pi C_0)(\Pi C_1)^{-1}$ .

A.34.





A.35.



A.36. We have  $[a, b] \in \mathbb{Z}_p$  and  $[a^{k-1}, G] \subseteq \mathbb{Z}_p$  (since  $k-1$  is even). Therefore

$$\Pi C = a^{k-2} b a^{-(k-2)} c a^{k-1} c^{-1} b^{-1} c a^{-(k-1)} c^{-1} \equiv b c c^{-1} b^{-1} c c^{-1} = e \pmod{\mathbb{Z}_p}$$

and

$$\Pi C' = a^{k-1} b a^{-(k-1)} c a^{k-1} b^{-1} a^{-(k-1)} c^{-1} \equiv b c b^{-1} c^{-1} = [b^{-1}, c^{-1}] \equiv [a^k, c] \not\equiv e \pmod{\mathbb{Z}_p}.$$

So  $(\Pi C)^{-1}(\Pi C') \notin \mathbb{Z}_p$ .

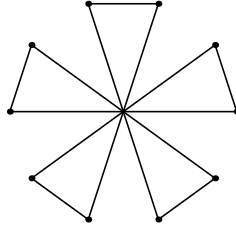
A.37. Since  $\bar{b} \in \langle \bar{a} \rangle$ , we see from the contrapositive of Corollary 3.8 that  $\mathbb{Z}_2 \not\subseteq \langle [a, b] \rangle$ . Therefore, we must have  $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$ . On the other hand, since  $\langle \bar{a}, \bar{c} \rangle = \bar{G}$ , but  $\{a, c\} \neq S$ , we see from the contrapositive of Corollary 3.5 that  $\mathbb{Z}_p \not\subseteq \langle [a, c] \rangle$ . Therefore  $[a, c]$  is the nontrivial element of  $\mathbb{Z}_2$ .

A.38. Suppose  $c$  neither centralizes  $G'$  nor inverts  $G'$ . Since  $c$  does not centralize  $G'$ , we know that it does not centralize  $\mathbb{Z}_p$ , so there exists  $x \in \{a, b\}$ , such that  $\mathbb{Z}_p \subseteq \langle [c, x] \rangle$ .

If  $\bar{x} \in \langle \bar{c} \rangle$ , then we may apply Case 6.1 with  $c$  and  $x$  in the roles of  $a$  and  $b$ .

Suppose, now, that  $\bar{x} \notin \langle \bar{c} \rangle$ . Since  $c$  neither centralizes nor inverts  $G'$ , we know  $|\bar{c}| > 2$ . So one of the cases of Section 5 applies with  $c$  and  $x$  in the roles of  $a$  and  $b$ .

**A.39.** The path  $(a, b)^k$  is a hamiltonian cycle in  $\langle \bar{a} \rangle$ :



Removing a single edge from this hamiltonian cycle yields the hamiltonian path  $L$ .

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**A.40.** Since  $k+1$  is even (and  $a$  inverts  $G'$ ), we know that  $a^{k+1}$  centralizes  $\gamma$ . So

$$(ab)^k = (a(a^k\gamma))^k = (a^{k+1}\gamma)^k = a^{k(k+1)}\gamma^k = (a^m)^{(k+1)/2}\gamma^k = e^{(k+1)/2}\gamma^k = \gamma^k.$$


---

**A.41.** If  $p \mid k$ , then

$$\Pi L = (ab)^k b^{-1} = \gamma^k \cdot \gamma^{-1} a^{-k} = \gamma^{k-1} a^{-k}.$$

If  $p \nmid k$ , then

$$\Pi L = (ab)^k a^{-1} = \gamma^k a^{-1}.$$


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**A.42.** Let  $C_1$  be the hamiltonian cycle obtained from  $C_0$  by applying Lemma 2.12 to  $[\bar{e}](b, a, b)$  and  $[\bar{a}\bar{b}\bar{c}](a^{-1})$  (so  $s = b$ ,  $t = a$ ,  $u = c$ , and  $h = bc$ ). Then

$$((\Pi C_0)^{-1}(\Pi C_1))^{bc} = [t^{-1}, u][s, t^{-1}]^u = [a^{-1}, c][b, a^{-1}]^c.$$

Since  $b$  inverts  $G'$ , but  $c$  centralizes  $G'$ , this tells us

$$(\Pi C_0)^{-1}(\Pi C_1) = ([a^{-1}, c][b, a^{-1}])^{-1} = [c, a^{-1}][a^{-1}, b].$$

Also, since  $C_2$  is obtained from  $C_1$  by applying Lemma 2.12 to the path  $[\bar{a}^2](b, a, b)$  and the edge  $[\bar{a}^3\bar{b}\bar{c}](a^{-1})$  (so  $s = b$ ,  $t = a$ ,  $u = c$ , and  $h = a^2bc$ ), we have

$$((\Pi C_1)^{-1}(\Pi C_2))^{a^2bc} = [t^{-1}, u][s, t^{-1}]^u = [a^{-1}, c][b, a^{-1}]^c.$$

Since  $a$  and  $b$  invert  $G'$ , but  $c$  centralizes  $G'$ , this tells us

$$(\Pi C_1)^{-1}(\Pi C_2) = ([a^{-1}, c][b, a^{-1}])^{-1} = [c, a^{-1}][a^{-1}, b].$$

Putting these two calculations together tells us

$$\begin{aligned} (\Pi C_0)^{-1}(\Pi C_2) &= ((\Pi C_0)^{-1}(\Pi C_1))((\Pi C_1)^{-1}(\Pi C_2)) \\ &= ([c, a^{-1}][a^{-1}, b])([c, a^{-1}][a^{-1}, b]) \\ &= [c, a^{-1}]^2[a^{-1}, b]^2 \\ &= [a^{-1}, b]^2 \end{aligned} \quad ([a, c] \in \mathbb{Z}_2).$$

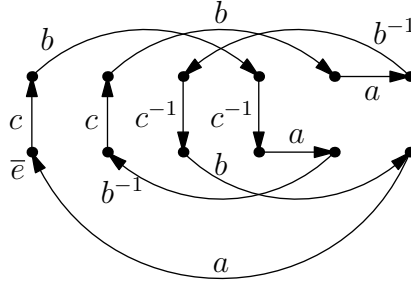

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**A.43.** Since  $c$  centralizes  $G'$ , the map  $x \mapsto [x, c]$  is a homomorphism to  $\mathbb{Z}_2$  whose kernel contains  $G'$ . Therefore

$$\begin{aligned}
 \Pi C_0 &= ((ba)^k a^{-1}) c ((ba)^k a^{-1})^{-1} c^{-1} \\
 &= [((ba)^k a^{-1})^{-1}, c^{-1}] \\
 &= [a^{k(k+1)-1} \gamma^k, c^{-1}] \\
 &= [a^{-1}, c^{-1}] && (k(k+1) \text{ is even and } [\gamma, c] = e) \\
 &= [a, c].
 \end{aligned}$$


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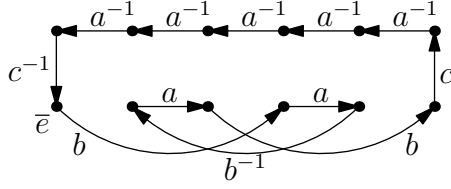
**A.44.**



**A.45.** Let  $z$  be the nontrivial element of  $\mathbb{Z}_2$ , so  $[a, c] = [b, c] = z$ . Then

$$\begin{aligned}
 \Pi C &= cbc^{-1} \cdot ab^{-1} \cdot cbab^{-1}c^{-1} \cdot ba \\
 &= {}^c b \cdot ab^{-1} \cdot {}^c (bab^{-1}) \cdot ba \\
 &= bz \cdot ab^{-1} \cdot (bab^{-1})z \cdot ba \\
 &= b \cdot ab^{-1} \cdot (bab^{-1}) \cdot ba \cdot z^2 && (\mathbb{Z}_2 \text{ is in the center of } G) \\
 &= ba^3 && (z^2 = e \text{ because } z \in \mathbb{Z}_2) \\
 &= (a^k \gamma) a^k && (k = 3) \\
 &= a^{2k} \gamma^{-1} && (a \text{ inverts } G \text{ and } k = 3 \text{ is odd}) \\
 &= \gamma^{-1} && (a^{2k} = a^m = e).
 \end{aligned}$$


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**A.46.****A.47.** We have

$$\begin{aligned}
 \Pi C &= (bab^{-1}ab)(ca^{-5}c^{-1}) \\
 &= (bab^{-1}aba)(a^{-1}cac^{-1}) && (a^6 = a^{2k} = e, \text{ so } a^{-5} = a) \\
 &= (bab^{-1}aba)[a, c^{-1}].
 \end{aligned}$$

Since  $[a, c] \in \mathbb{Z}_2$ , we have  $[a, c^{-1}] = [a, c]$ . Also, we have

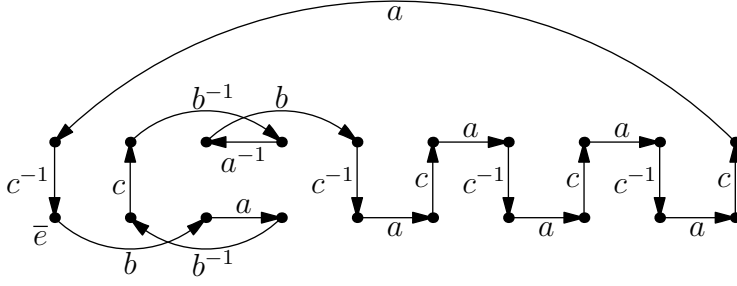
$$\begin{aligned}
 bab^{-1}aba &= (a^3\gamma)a(\gamma^{-1}a^{-3})a(a^3\gamma)a \\
 &= a^{3+1-3+1+3+1}\gamma^{-3} && (a \text{ inverts } \gamma) \\
 &= \gamma^{-3} && (a^6 = a^{2k} = e).
 \end{aligned}$$

Therefore  $\Pi C = \gamma^{-3}[a, c]$ .

**A.48.** Let  $z = [a, c]$  be the nontrivial element of  $\mathbb{Z}_2$ . Then every element of  $\widehat{G} = G/\mathbb{Z}_p$  can be written uniquely in the form  $a^i c^j w^k$  with  $0 \leq i \leq 5$  and  $j, k \in \{0, 1\}$ . The hamiltonian cycle  $C$  visits the vertices of  $\text{Cay}(\widehat{G}; S)$  in the following order:

$$\begin{array}{ccccccc}
 & e & \xrightarrow{a} & a & \xrightarrow{a} & a^2 & \xrightarrow{c} & a^2 c \\
 \xrightarrow{a} & a^3 c z & \xrightarrow{a} & a^4 c & \xrightarrow{a} & a^5 c z & \xrightarrow{a} & c \\
 \xrightarrow{a} & a c z & \xrightarrow{c} & a z & \xrightarrow{a^{-1}} & z & \xrightarrow{a^{-1}} & a^5 z \\
 \xrightarrow{b} & a^2 z & \xrightarrow{a} & a^3 z & \xrightarrow{a} & a^4 z & \xrightarrow{c} & a^4 c z \\
 \xrightarrow{a^{-1}} & a^3 c & \xrightarrow{a^{-1}} & a^2 c z & \xrightarrow{a^{-1}} & a c & \xrightarrow{a^{-1}} & c z \\
 \xrightarrow{a^{-1}} & a^5 c & \xrightarrow{c} & a^5 & \xrightarrow{a^{-1}} & a^4 & \xrightarrow{a^{-1}} & a^3 \\
 \xrightarrow{b} & e & & & & & & 
 \end{array}$$

A.49.



A.50. Lemma 2.12 tells us

$$((\Pi C_0)^{-1}(\Pi C_1))^{b^2} = [u, t^{-1}][s, t^{-1}]^u = [b, a^{-1}][b, a^{-1}]^b.$$

Since  $b$  centralizes  $G'$ , we have

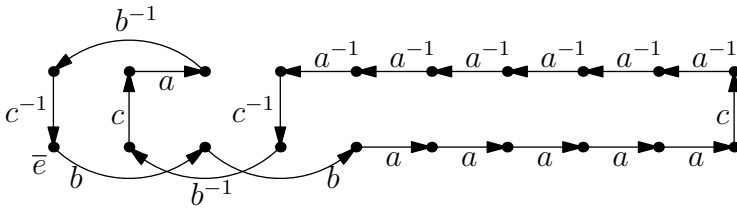
$$((\Pi C_0)^{-1}(\Pi C_1))^{b^2} = (\Pi C_0)^{-1}(\Pi C_1)$$

and

$$[b, a^{-1}]^b = [b, a^{-1}] = [b, a]^{-1} = [a, b].$$

Therefore  $(\Pi C_0)^{-1}(\Pi C_1) = [a, b]^2$ .

A.51.

A.52. Calculating modulo  $\mathbb{Z}_p$ , we have

$$\Pi C = b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1})$$

$$\equiv b^2(a^{-1})(b^{-1}ab^{-1}[a, c])$$

$$\equiv [a, c]$$

$$\left( \begin{array}{l} m-4 \text{ is even, but } 1-k = -1 \text{ is odd,} \\ \text{so } c \text{ commutes with } a^{-(m-4)} \\ \text{but not with } ab^{-1} \pmod{\mathbb{Z}_p} \end{array} \right)$$

$$(a \text{ commutes with } b \pmod{\mathbb{Z}_p}).$$

**A.53.** Calculating modulo  $\mathbb{Z}_2$ , we have

$$\begin{aligned}
 \Pi C &= b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \\
 &= (a^2\gamma)^2(a^{m-5}ca^{-(m-4)}c^{-1})((\gamma^{-1}a^{-2})ca(\gamma^{-1}a^{-2})c^{-1}) \quad (b = a^k\gamma = a^2\gamma) \\
 &\equiv \gamma^2 a^{-1} \gamma^{-1} ca \gamma^{-1} c^{-1} \quad \left( \begin{array}{l} a^2 \text{ commutes with } \gamma \text{ and} \\ a \text{ commutes with } c \pmod{\mathbb{Z}_2} \end{array} \right) \\
 &\equiv \gamma^3 \cdot {}^c(\gamma^{-1}) \quad \left( \begin{array}{l} a \text{ inverts } \gamma \text{ and} \\ \text{commutes with } c \pmod{\mathbb{Z}_2} \end{array} \right) \\
 &= \gamma^3 \cdot \gamma^{\pm 1} \quad (c \text{ either centralizes or inverts } G') \\
 &\in \{\gamma^2, \gamma^4\}.
 \end{aligned}$$


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**A.54.** The main result of [1] states that the generalized Petersen graph  $GP(m, k)$  has a hamiltonian cycle if  $\gcd(m, k) = 1$  and

$$\text{either } m \not\equiv 5 \pmod{6} \text{ or } k \notin \{2, (m-1)/2, (m+1)/2, m-2\}.$$

(More generally, see [A, Thm. 3] for a complete determination of which generalized Petersen graphs have hamiltonian cycles.)

$\text{Cay}(G; a, b)$  is the generalized Petersen graph  $GP(2n, k)$ , where  $b^a = b^k$  and  $1 \leq k < 2n$ . We must have  $\gcd(2n, k) = 1$  (because  $b^k$ , like  $b$ , must generate  $\langle b \rangle$ ), and it is obvious that  $2n \not\equiv 5 \pmod{6}$  (since  $2n$  is even), so Bannai's theorem provides a hamiltonian cycle in  $\text{Cay}(G; a, b)$ .

## References

- [A] B. Alspach, The classification of Hamiltonian generalized Petersen graphs. J. Combin. Theory Ser. B 34 (1983), no. 3, 293312. [MR 0714452](#)  
[http://dx.doi.org/10.1016/0095-8956\(83\)90042-4](http://dx.doi.org/10.1016/0095-8956(83)90042-4)
- 

**A.55.** For the reader's convenience, we translate the calculations in the last paragraph of Case 1 of the proof of [3, Prop. 6.1] into our notation.

Let  $w$  be the nontrivial element of  $\mathbb{Z}_2$ . Then every element of  $G$  can be written uniquely in the form  $a^i b^j w^k$  with  $i, j, k \in \{0, 1\}$ . To see that  $((a, b)^4 \# b^{-1})$  is a hamiltonian cycle, note that it visits the vertices of  $\text{Cay}(G, a, b)$  in the following order:

$$\underline{e}, \underline{a}, \underline{ab}, \underline{bw}, \underline{w}, \underline{aw}, \underline{abw}, \underline{b}, \underline{e}.$$

Since

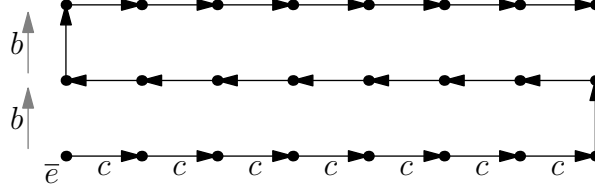
$$(ab)^2 = b(b^{-1}aba)b = b[b, a]b = [a, b]^{-1}b^2 = (-1, -2, 0) \cdot (0, 2, 2) = (-1, 0, 2),$$

the voltage of this hamiltonian cycle is

$$(ab)^4 b^{-2} = ((ab)^2)^2 b^{-2} = (-1, 0, 2)^2 \cdot b^{-2} = (0, 0, 4) \cdot (0, -2, -2) = (0, -2, 2).$$


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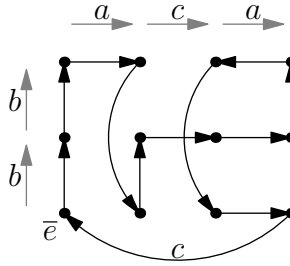
A.56.

A.57. Calculating modulo  $\mathbb{Z}_p$ , we have

$$\begin{aligned}
 \Pi C &= (c^{\ell-1}bc^{-(\ell-1)}bc^{\ell-1})a(c^{\ell-1}bc^{-(\ell-1)}bc^{\ell-1})^{-1}a \\
 &= [(c^{\ell-1}bc^{-(\ell-1)}bc^{\ell-1})^{-1}, a] \\
 &\equiv [c, a]^{-(\ell-1)}[b, a]^{-1}[c, a]^{\ell-1}[b, a]^{-1}[c, a]^{-(\ell-1)} \quad (G'/\mathbb{Z}_p \text{ is in the center of } G/\mathbb{Z}_p) \\
 &\equiv [c, a][b, a][c, a][b, a][c, a] \quad (\ell-1 \text{ and } -1 \text{ are odd}) \\
 &\equiv [c, a].
 \end{aligned}$$

This is nontrivial (mod  $\mathbb{Z}_p$ ), so  $\Pi C$  projects nontrivially to  $\mathbb{Z}_2$ .

A.58.

A.59. Calculating modulo  $\mathbb{Z}_p$ , we have

$$\begin{aligned}
 \Pi C_0 &= b^2ab^2cababac \\
 &= b^6acac \quad ([a, b] \text{ and } [b, c] \text{ are in } \mathbb{Z}_p, \text{ and } a^2 = e) \\
 &= e \cdot [a, c] \quad (|\bar{b}| = 3 \Rightarrow b^3 \in \mathbb{Z}_2 \pmod{\mathbb{Z}_p}) \\
 &= [a, c].
 \end{aligned}$$

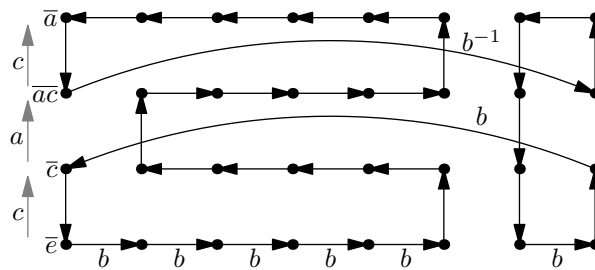
This is nontrivial (mod  $\mathbb{Z}_p$ ).

$$\begin{aligned}
\Pi C &= (abab^{n-2}ab^{-(n-3)})c(abab^{n-2}ab^{-(n-3)})^{-1}c^{-1} \\
&= [(abab^{n-2}ab^{-(n-3)})^{-1}, c^{-1}] \\
&\equiv [abab^{n-2}ab^{-(n-3)}, c] && \text{(Lemma 3.6 implies } [x^i, y^j] \equiv [x, y]^{ij} \text{)} \\
&\equiv [a, c] [b, c] [a, c] [b, c]^{n-2} [a, c] [b, c]^{-(n-3)} && \text{(Lemma 3.6)} \\
&= [a, c]^3 [b, c]^2 && (G' \text{ is abelian)} \\
&\equiv [a, c] && (3 \text{ is odd and } 2 \text{ is even).}
\end{aligned}$$
$$\Pi C' \equiv [b^{-1}, c][a, b^{-1}]^c \equiv [b, c][a, b] \pmod{\mathbb{Z}_p}.$$

By assumption, this projects nontrivially to  $\mathbb{Z}_2$ .

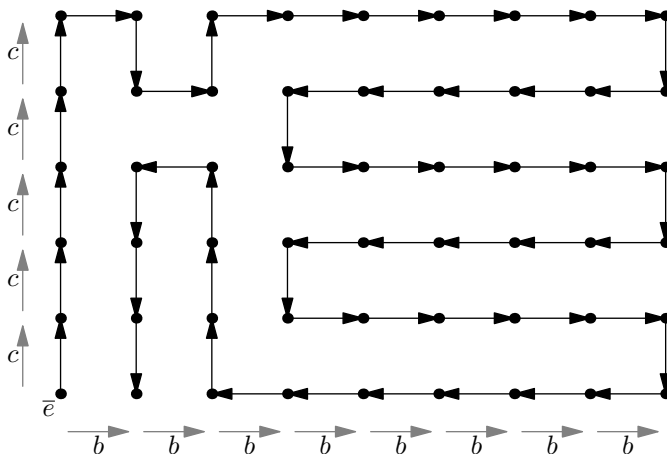


A.64.

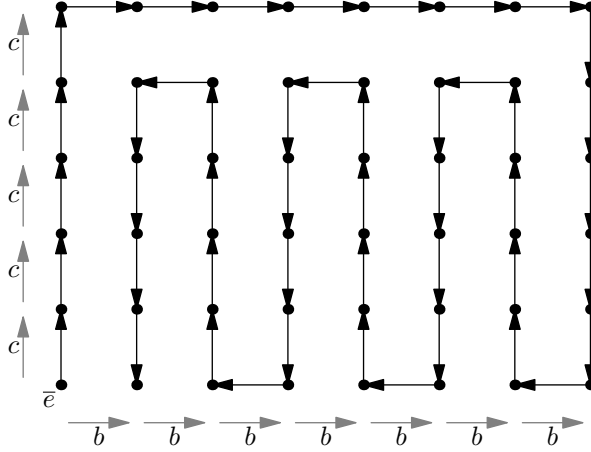

 A.65. Write  $a^{cb} = a\gamma$ , with  $\gamma \in G'$ . Then

$$(b^{-1}c)^2 a (cb)^2 = (cb)^{-2} a (cb)^2 = a^{(cb)^2} = (a^{cb})^{cb} = (a\gamma)^{cb} = a^{cb} \gamma^{cb} = (a\gamma) \gamma^{-1} = a.$$

A.66.



A.67.



**A.68.** Let  $C'$  be the hamiltonian cycle obtained by applying the  $a$ -transform. To show that the  $a$ -transform multiplies the voltage by  $\gamma_a$ , we wish to show

$$(\Pi C)^{-1}(\Pi C') = [a, b^{-1}][c, a].$$

Lemma 2.12 tells us

$$((\Pi C)^{-1}(\Pi C'))^{ab} = [t^{-1}, u][s, t^{-1}]^u = [a^{-1}, b][c^{-1}, a^{-1}]^b = [a, b][c^{-1}, a]^b,$$

because  $a^{-1} = a$ . So

$$(\Pi C)^{-1}(\Pi C') = ([a, b][c^{-1}, a]^b)^{(ab)^{-1}} = [a, b]^{b^{-1}a} [c^{-1}, a]^a = [a, b^{-1}][c, a],$$

because

$$[a, b]^{b^{-1}a} = (ab)(ab^{-1}ab)(b^{-1}a) = abab^{-1} = [a, b^{-1}]$$

and

$$\begin{aligned} [c^{-1}, a]^a &= [a, c^{-1}] && (a \text{ inverts } G') \\ &= [a, c]^{-1} && (c \text{ centralizes } G') \\ &= [c, a]. \end{aligned}$$

**A.69.** Lemma 2.12 tells us that the  $b$ -transform multiplies the voltage by a conjugate of

$$[t^{-1}, u][s, t^{-1}]^u = [b, a][c^{-\varepsilon}, b]^a = [b, a][b, c^{-\varepsilon}] = \gamma_b.$$

**A.70.** Let  $z$  be the nontrivial element of  $\mathbb{Z}_2$ , and write  $L = (s_i)_{i=1}^r$ , so  $s_i \in \{b^{\pm 1}, c^{\pm 1}\}$  and  $r$  is odd. Then, calculating mod  $\mathbb{Z}_p$ , we have  $[s_i, a] \equiv z$  for all  $i$ , so

$$\Pi C = \left(\prod_{i=1}^r s_i\right) a \left(\prod_{i=1}^r s_i\right)^{-1} a = \left[\left(\prod_{i=1}^r s_i\right)^{-1}, a\right] \equiv \prod_{i=1}^r [s_i, a]^{-1} \equiv \prod_{i=1}^r z = z^r = z.$$


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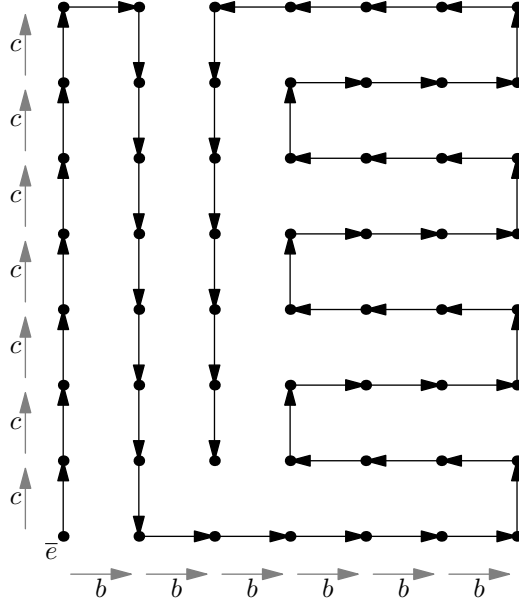
**A.71.** Since  $b$  and  $c$  centralize  $G'$ , Lemma 3.6 implies that if we let  $\varphi(x) = [x, a]$ , then  $\varphi$  is a homomorphism from  $\langle b, c \rangle$  to  $G'$ . Note that, since the image of  $\varphi$  is a subgroup of the abelian group  $G'$ , the kernel of  $\varphi$  must contain  $\langle b, c \rangle'$ .

Write  $L = (s_i)_{i=1}^r$ . Then  $\Pi C = [(\prod_{i=1}^r s_i)^{-1}, a]$ . Since the sum of the exponents of the occurrences of  $b$  in  $L$  is 1, and the sum of the exponents of the occurrences of  $c$  is 0, we know  $\prod_{i=1}^r s_i \equiv b \pmod{\langle b, c \rangle'}$ . So the preceding paragraph tells us that

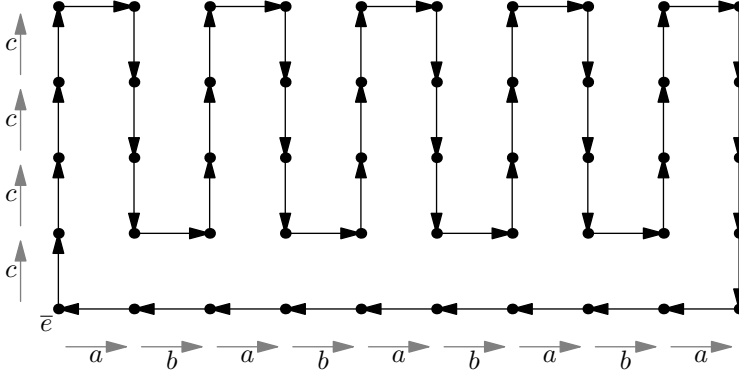
$$\left[\left(\prod_{i=1}^r s_i\right)^{-1}, a\right] = [b, a]^{-1} = [a, b].$$


---

**A.72.**



A.73.

A.74. Suppose  $\bar{a} \in \langle \bar{b} \rangle$ .

If  $b$  does not centralize  $G'$ , then Assumption 9.1 tells us that  $|\bar{b}| = 2$ , so we must have  $\bar{a} = \bar{b}$ , which contradicts the assumption that Case 4.1 does not apply.

We now know that  $b$  centralizes  $G'$ . Since  $G'$  is abelian (indeed, it is cyclic), this implies that  $\langle b, G' \rangle$  is abelian. However, the fact that  $\bar{a} \in \langle \bar{b} \rangle$  means that  $a \in \langle b, G' \rangle$ . Since  $\langle b, G' \rangle$  is abelian, this implies that  $a$  centralizes  $G'$ . This contradicts (3.3A).

A.75. Suppose  $s, t \in S$ , such that  $s$  commutes with  $t$  (and  $s \neq t$ ). There exist  $x, y \in S$ , such that  $\mathbb{Z}_2 \subseteq \langle [x, y] \rangle$ . Since  $\{s, t\} \neq \{x, y\}$ , we may assume  $s \notin \{x, y\}$  (after interchanging  $s$  and  $t$  if necessary). Since  $t$  does not centralize  $G'$ , there exists  $u \in S$ , such that  $\mathbb{Z}_p \subseteq \langle [t, u] \rangle$ . Then  $\{x, y, t, u\} \subseteq S \setminus \{s\}$ , so

$$G' = \langle \mathbb{Z}_2, \mathbb{Z}_p \rangle \subseteq \langle [x, y], [t, u] \rangle \subseteq \langle S \setminus \{s\} \rangle,$$

so  $\langle S \setminus \{s\} \rangle \triangleleft G$ . Therefore Lemma 2.13 applies

A.76. Note that  $L_0 = (c, a, c)$  is a hamiltonian path in  $\text{Cay}(\langle \bar{a}, \bar{c} \rangle; \{a, c\})$  (because we have  $\langle \bar{a}, \bar{c} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ ). Therefore

$$L_1 = (c, a, c, b)^2 \# = (L_0, b, L_0^{-1})$$

is a hamiltonian path in  $\text{Cay}(\langle \bar{a}, \bar{b}, \bar{c} \rangle; \{a, b, c\})$ . So

$$C = ((c, a, c, b)^2 \#, d)^2 = (L_1, d, L_1^{-1}, d)$$

is a hamiltonian cycle in  $\text{Cay}(\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle; \{a, b, c, d\})$ .

**A.77.** Lemma 2.12 tells us

$$((\Pi C)^{-1}(\Pi C'))^{bc} = [t^{-1}, u] [s, t^{-1}]^u = [a, b] [c, a]^b = [a, b] [a, c] = \gamma.$$

Since  $b$  and  $c$  both invert  $G'$ , we know that  $bc$  centralizes  $G'$ , so  $(\Pi C)^{-1}(\Pi C') = \gamma$ .

---

**A.78.** This is exactly the same calculation as in A.77, except that  $(\Pi C')^{-1}(\Pi C'')$  is conjugated by  $cd$ , instead of  $bc$ . Since  $cd$ , like  $bc$ , centralizes  $G'$ , this change does not affect the result of the calculation at all, so, as before, the voltage is multiplied by  $\gamma$ .

---

**A.79.** Note that

- if  $y$  centralizes  $G'$ , then  $[xy, z] = [x, z] [y, z]$  and  $[y^{-1}, z] = [y, z]^{-1}$ , but
- if  $y$  inverts  $G'$ , then  $[xy, z] = [x, z]^{-1} [y, z]$  and  $[y^{-1}, z] = [y, z]$ .

Therefore

$$\begin{aligned} \Pi C &= ((cacb)^2 b^{-1} d)^2 \\ &= [(cacb)^2 b^{-1}, d] \\ &= [(cacb)^2, d]^{-1} [b, d] && (b \text{ inverts } G') \\ &= ([ca, d]^2 [cb, d]^2)^{-1} [b, d] && (cb \text{ and } cb \text{ centralize } G') \\ &= \left( ([c, d]^{-1} [a, d])^2 ([c, d]^{-1} [b, d])^2 \right)^{-1} [b, d] && (a \text{ and } b \text{ invert } G') \\ &= [c, d]^4 [a, d]^{-2} [b, d]^{-1} \\ &= [c, d]^4 [d, a]^2 [d, b]. \end{aligned}$$


---

**A.80.** We have  $[d, a]^6 [d, b] \in \mathbb{Z}_2$ . Assuming the same is true when we interchange  $a$  and  $c$ , we also have  $[d, c]^6 [d, b] \in \mathbb{Z}_2$ . So

$$[d, c]^6 \equiv [d, b]^{-1} \equiv [d, a]^6 \pmod{\mathbb{Z}_2}.$$

Since  $p \neq 3$ , this implies  $[d, c] \equiv [d, a] \pmod{\mathbb{Z}_2}$ .

---

**A.81.** Suppose  $[s, a] \notin \mathbb{Z}_2$ , so  $g = s$ . By the definition of  $s$ , we also know  $[s, a] \notin \mathbb{Z}_p$ . Therefore  $[s, a]$  generates  $G'$ .

Assume now that  $[s, a] \in \mathbb{Z}_2$ , so  $g = sb^2$ . Calculating modulo  $\mathbb{Z}_p$ , we have

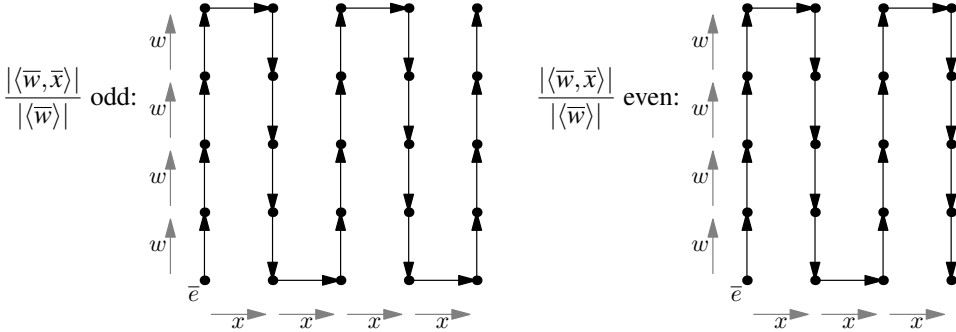
$$\begin{aligned} [sb^2, a] &= [s, a][b, a]^2 && \text{(by Lemma 3.6, since } G' \text{ is central in } G/\mathbb{Z}_p) \\ &\equiv [s, a] \\ &\notin \mathbb{Z}_p && \text{(definition of } s). \end{aligned}$$

Calculating modulo  $\mathbb{Z}_2$ , we have

$$\begin{aligned} [sb^2, a] &\equiv [b^2, a] && ([s, a] \in \mathbb{Z}_2) \\ &\neq e. \end{aligned}$$

so  $[sb^2, a]$  generates  $G'$ .

**A.82.** Let  $L$  be the following hamiltonian path in  $\text{Cay}(\langle \bar{w}, \bar{x} \rangle; w, x)$ :



Note that  $L$  contains the oriented path  $[\bar{w}^{|\bar{w}|-2}](w, x, w^{-1})$ . We can also write this path as  $[hw^{-1}y^{-1}](w, x, w^{-1})$ , for  $h = w^{|\bar{w}|-1}y$ .

Then  $C' = (L, y, L^{-1}, y^{-1})$  is a cycle through the vertices of  $\text{Cay}(\langle \bar{w}, \bar{x}, \bar{y} \rangle; w, x, y)$  that are in  $\langle \bar{w}, \bar{x} \rangle \cup \bar{y}\langle \bar{w}, \bar{x} \rangle$ . Therefore, if  $C''$  is any hamiltonian cycle in  $\text{Cay}(\langle \bar{w}, \bar{x} \rangle; w, x)$  that shares an edge with  $C'$ , then an appropriate connected sum

$$C_0 = C' \#_{t_1}^{s_1} - g_1 C'' \#_{t_2}^{s_2} g_2 C'' \#_{t_3}^{s_3} \cdots \#_{t_d}^{s_d} \pm g_d C''$$

of  $C'$  with translates of  $C''$  is a hamiltonian cycle in  $\text{Cay}(\bar{G}; S_0)$ .

Note that  $C'$  contains both  $[hw^{-1}y^{-1}](w, x, w^{-1})$  and  $[\bar{h}\bar{x}](x^{-1})$ . Therefore, if the edge used to form the first connected sum  $C' \#_{t_1}^{s_1} - g_1 C''$  is not in either of these paths, then  $C_0$  also contains both of these paths. (For example, we could use the edge  $[\bar{y}\bar{w}](w^{-1})$  to form the connected sum.)

**A.83.** It suffices to show that  $C_0$  has an edge labeled  $s^{\pm 1}$  that is not in either of the given paths, for we may assume that this oriented edge is of the form  $[s^{\varepsilon}](s^{-\varepsilon})$  (by replacing  $C_0$  with a translate).

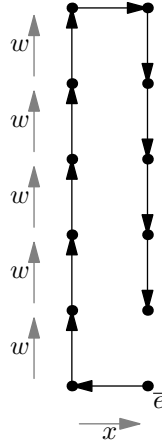
If  $s \neq b$ , then Note 9.3 implies there is at least one edge labeled  $s^{\pm 1}$ . If  $s \notin \{w^{\pm 1}, x^{\pm 1}\}$ , then this edge is obviously not in either of the given paths (since all of the edges in those paths are labeled  $w^{\pm 1}, x^{\pm 1}$ ).

Therefore, we may assume  $s \in \{w^{\pm 1}, x^{\pm 1}, b\}$ . To deal with this situation, we assume  $C_0$  has been constructed as in A.82 (from the path  $L$  and cycle  $C'$  that are specified there).

Suppose  $s = w^{\pm 1}$ . The edges  $[\overline{wy}](w^{-1})$  and  $[\overline{xy}](w)$  are in  $C'$ , but are not in either of the given paths. One of these edges may have been removed in forming the connected sum that defines  $C_0$ , but the other will remain. So  $C_0$  has at least one edge labeled  $w^{\pm 1}$ .

Suppose  $s = x^{\pm 1}$ .

- If  $|\langle \overline{w}, \overline{x} \rangle / \langle \overline{w} \rangle| > 2$ , then  $C'$  contains the edges  $[\overline{x}](x)$  and  $[\overline{x^2y}](x^{-1})$ . At least one of these must be in  $C_0$  (and neither of these edges is in the given paths).
- If  $|\overline{G} : \langle \overline{w}, \overline{x} \rangle| > 2$ , then  $C_0 \neq C'$  (in other words,  $d > 1$  in the definition of  $C_0$ ). We may assume  $C'$  has at least one edge labeled  $x^{\pm 1}$ , and that only edges labeled  $w^{\pm 1}$  are used in forming the connected sums. Therefore, this edge labeled  $x^{\pm 1}$  is in  $C_0$  (and it is not in the given paths).
- We may now assume  $|\overline{G} : \langle \overline{w} \rangle| = 4$ . Since (9.5B) tells us that  $|\overline{G}| > 16$  (and we know  $|\overline{a}| = 2$ ), we must have  $|\overline{w}| > 2$ . Therefore, we have the following hamiltonian path  $\tilde{L}$  in  $\text{Cay}(\langle \overline{w}, \overline{x} \rangle; w, x)$ :



Use this path in the place of  $L$  to construct cycles  $\tilde{C}'$  and  $\tilde{C}_0$  analogous to  $C'$  and  $C_0$  (and let  $h = w^{|\overline{w}|-1}x^{-1}y$ ). Note that  $\tilde{L}$  has two edges labelled  $x^{\pm 1}$ , so  $\tilde{C}'$  has four edges labelled  $x^{\pm 1}$ . Two of these are in the given paths, and one may be deleted in the construction of the connected sum, but at least one of these edges labelled  $x^{\pm 1}$  remains in  $\tilde{C}_0$  and is not in either of the given paths.

We may now assume  $s = b \notin \{w, x\}$ . From Note 9.3, we know that  $w \notin \langle \overline{b} \rangle$ . Therefore, it is easy to construct a hamiltonian path  $P = (t_i)_{i=1}^r$  in  $\text{Cay}(\langle \overline{w}, \overline{b} \rangle : b, w)$  that has at least one edge labeled  $b^{\pm 1}$ , and such that  $t_r = w$ . Now, in place of the hamiltonian path

$$L = (w^r, x, w^{-r}, x, w^r, x, w^{-r}, x, \dots)$$

that was used in A.82, use the hamiltonian path

$$\tilde{L} = (P, x, P^{-1}, x, P, x, P^{-1}, x, \dots)$$

to construct cycles  $\tilde{C}'$  and  $\tilde{C}_0$  analogous to  $C'$  and  $C_0$  (and let  $h = y \prod_{i=1}^r t_i$ ). Then  $C_0$  contains at least two edges labeled  $b^{\pm 1}$  (one from the first occurrence of  $P$  in  $L$ , and another from the first occurrence of  $P^{-1}$ ). Neither of these is in the given paths (because  $b \notin \{w, x\}$ ) and at least one of them remains in  $\tilde{C}_0$ .

---

**A.84.** Suppose  $a \in \langle S_0, G' \rangle$ . Then

$$G = \langle S \rangle = \langle S_0, a \rangle \subseteq \langle S_0, G' \rangle = \langle S_0, \mathbb{Z}_p, \mathbb{Z}_2 \rangle \subseteq \langle S_0, \langle S_0 \rangle', \mathbb{Z}_2 \rangle = \langle S_0, \mathbb{Z}_2 \rangle,$$

so Lemma 3.4 implies  $G = \langle S_0 \rangle$ . This contradicts the irredundance of  $S$ .

---

**A.85.** If  $b \notin \{x^{\pm 1}, t^{\pm 1}\}$ , then Note 9.3 immediately implies that (9.5A) is satisfied. Then, since  $b \notin \{t^{\pm 1}\} = \{y^{\pm 1}\}$ , we may assume  $b \in \{x^{\pm 1}\}$ . We wish to show  $\langle \bar{w} \rangle \not\subseteq \langle \bar{w}, \bar{b} \rangle$ . In other words, we wish to show  $\bar{b} \notin \langle \bar{w} \rangle$ .

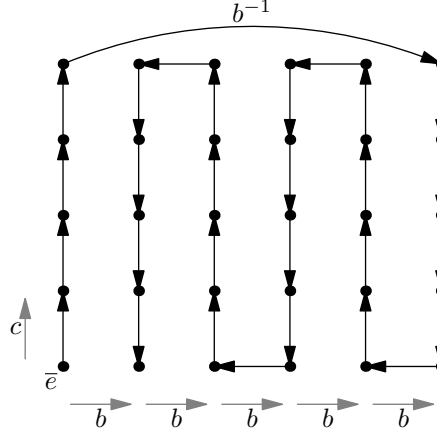
Suppose  $\bar{b} \in \langle \bar{w} \rangle$ . Then, since  $b$  does not centralize  $G'$ , we know that  $w$  does not centralize  $G'$ , so Assumption 9.1 tells us that  $|\bar{w}| = 2$ . Since  $\bar{b} \in \langle \bar{w} \rangle$ , this implies that  $\bar{b} = \bar{w}$ , so Case 4.1 applies.

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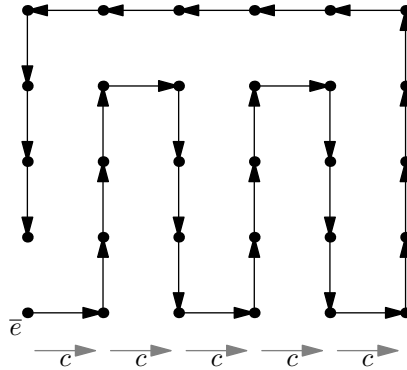


**A.86.** Since  $\mathbb{Z}_p \subseteq \langle [a, b] \rangle$  and  $b$  centralizes  $G'$ , we know  $|\bar{b}|$  is divisible by  $p$ , so  $|\bar{b}| > 2$ .

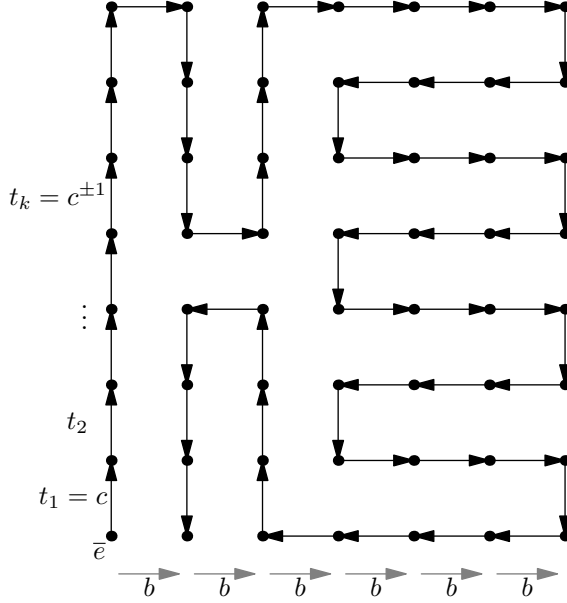
If  $|\bar{b}|$  is even, we may let  $L$  be a hamiltonian path of the following shape:



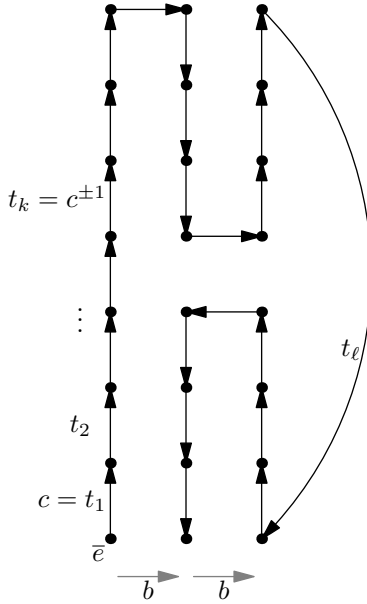
Henceforth, we assume that  $|\bar{b}|$  is odd. From Corollary 3.8, we see that  $|\overline{G} : \langle \bar{a}, \bar{b}, \bar{c} \rangle|$  is even (so, in particular,  $\langle \bar{a}, \bar{b}, \bar{c} \rangle \neq \overline{G}$ ). And Corollary 3.8 also implies that  $|\langle \bar{a}, \bar{b}, \bar{c} \rangle : \langle \bar{a}, \bar{b} \rangle|$  is even. Let  $(t_i)_{i=1}^{\ell-1}$  be a hamiltonian path in  $\text{Cay}(\overline{G}/\langle \bar{a}, \bar{b} \rangle; S \setminus \{a, b\})$ , such that  $t_1 = c$  and  $t_k = c^{\pm 1}$  for some  $k \notin \{1, 2\}$ . For example,  $(t_i)_{i=1}^{\ell-1}$  could be of the following form:



If  $|\bar{b}| > 3$ , we may let  $L$  be a hamiltonian path of the following shape:



Now assume  $|\bar{b}| = 3$ . The hamiltonian path  $(t_i)_{i=1}^{\ell-1}$  that is pictured above can be extended to a hamiltonian cycle  $(t_i)_{i=1}^{\ell}$ , such that  $\prod_{i=1}^{\ell} t_i \in G'$ . Now, we may let  $L$  be the following hamiltonian path:



**A.87.** Lemma 2.12 tells us that

$$((\Pi C)^{-1}(\Pi C'))^{ab} = [t^{-1}, u][s, t^{-1}]^u = [a, b][c^{-1}, a]^b.$$

Since  $a$  inverts  $G'$  and  $b$  centralizes  $G'$ , this implies

$$(\Pi C)^{-1}(\Pi C') = [b, a][a, c^{-1}] = \gamma_a.$$

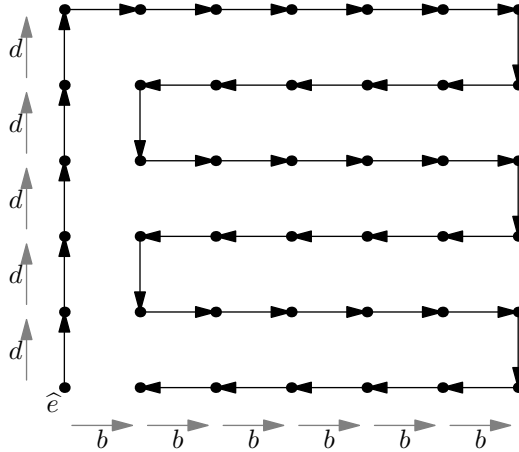
**A.88.** Lemma 2.12 tells us that the  $b$ -transform multiplies the voltage by a conjugate of

$$\begin{aligned} [t^{-1}, u][s, t^{-1}]^u &= [b^{-\delta}, a][c^{-\varepsilon}, b^{-\delta}]^a \\ &= [b^{-\delta}, a][b^{-\delta} \cdot c^{-\varepsilon}] \\ &= ([b, a][b \cdot c^{-\varepsilon}])^{-\delta} \\ &= ([a, b], [c^{-\varepsilon}, b])^{\delta} \\ &= (\gamma_b)^{\delta}. \end{aligned}$$

Since  $a$  inverts  $G'$ , and  $\delta \in \{\pm 1\}$ , this is conjugate to  $\gamma_b$ .

**A.89.** Let  $\widehat{G} = \overline{G}/\langle \bar{a} \rangle$ . We may let  $C_0 = (a, L, a, L^{-1})$ , where  $L = (t_i)_{i=1}^r$  is a hamiltonian path from  $\widehat{e}$  to  $\widehat{b}$  in  $\text{Cay}(\langle \widehat{S}_0 \rangle; b, d)$ , such that  $t_1 = d$  (and  $L$  contains at least one edge labeled  $b^{\pm 1}$ ).

Here is one way to construct such a hamiltonian path. We know that  $\mathbb{Z}_2 \not\subseteq \langle a, b, c \rangle'$  (since  $[s, t] \in \mathbb{Z}_p$  for all  $s \in \{a, b\}$  and  $t \in S$ ). On the other hand,  $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$  (from the choice of  $c$  and  $d$ ). So Corollary 3.8 tells us that  $|\langle \widehat{b}, \widehat{d} \rangle : \langle \widehat{b} \rangle|$  is even. (In fact,  $|\langle \widehat{b}, \widehat{c}, \widehat{d} \rangle : \langle \widehat{b}, \widehat{c} \rangle|$  is even.) Therefore, we may let  $L$  be a hamiltonian path of the following shape:



**A.90.** Lemma 2.12 tells us that the voltage is multiplied by a conjugate of

$$[t^{-1}, u] [s, t^{-1}]^u = [a, b] [d^{-1}, a]^b = [a, b] [d^{-1}, a] = \gamma.$$


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**A.91.** Let  $L = (t_i)_{i=1}^r$  be a hamiltonian path in  $\text{Cay}(\overline{H}/\langle \bar{a}, b \rangle; S \setminus \{a, b, c\})$ , such that  $t_1 = d$ . Then the desired hamiltonian cycle is  $(b, L, a, L^{-1}, b, L, a, L^{-1})$ .

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**A.92.** Lemma 2.12 tells us that the voltage is multiplied by a conjugate of

$$[t^{-1}, u] [s, t^{-1}]^u = [b, a] [d^{-1}, b]^a = [b, a] [b, d^{-1}],$$

since  $a$  inverts  $G'$ .

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